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ЗБОРНИК РАДОВА
ПРИРОДНО-МАТЕМАТИЧКОГ ФАКУЛТЕТА

11

СЕРИЈА ЗА МАТЕМАТИКУ

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EQUATION OF OSCILLATION OF A VISCOELASTIC BAR (II)

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INTRODUCTION

In paper |5| the following equation was analysed:

$$(1) \qquad \partial_t^2 u(t,x) - \partial_x^2 u(t,x) + \int_0^t \partial_t^2 u(t-\tau,x) G(\tau)\, d\tau = 0$$

with the initial conditions:

$$(2) \qquad (2') \; u(0,x) = 0, \qquad (2'') \; \partial_t u(0,x) = \delta(x)$$

where

$$(3) \qquad \{G(t)\} = \sum_{i=1}^{\infty} a_i \ell^{i\alpha} = \{ \sum_{i=1}^{\infty} a_i \frac{t^{i\alpha-1}}{\Gamma(i\alpha)} \}, \; a_1 > 0, \; 0 < \alpha < 1 .$$

The solution of equation (1) with condition (2´), which is a continuous function for $t \geq 0$, $x \neq 0$, having a line of symmetry $x = 0$, is:

$$(4) \qquad u(x) = \frac{1}{2} \ell \exp(-|x| \sum_{i=0}^{\infty} c_i \ell^{i\alpha-1}) \; ; \; c_0 = 1$$

the coefficients c_i can be obtained in the following way:

$$c_1 = \frac{1}{2} a_1$$

(5) $$c_2 = \frac{1}{2} (a_2 - (a_1/2)^2)$$

$$\cdots\cdots\cdots\cdots$$

$$c_i = \frac{1}{2} (a_i - \sum_{j=1}^{n-1} c_{i-j} c_j).$$

Solution (4) satisfies condition (2") too, if we have $i_1\alpha - 1 > 0$, where i_1 is the index of coefficients c_i such that $i_1 > 1$, $c_i = 0$ for $1 < i < i_1$ and $c_{i_1} \neq 0$.

This result is a generalization of the result from |3| in which problem (1), (2) was observed for:

(6) $$G(t) = 2\lambda \frac{t^{\alpha-1}}{\Gamma(\alpha)} + \lambda^2 \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + R(t), \quad \lambda > 0,\ 0 < \alpha < 1,$$

where either $R(t) \equiv 0$ or $R(0) = R'(0) = 0$ and $R''(t)$ is "sufficiently small" for $t \in [0,T]$, $R(t) \in C^\infty$.

This paper is a continuation of paper |5|. Our aim is to estimate the difference between the solution of equation (1) with conditions (2) when $G(t)$ is given by (6) and the solution when in the expression for $G(t)$ the remainder $R(t)$ is omitted (Proposition 3.). In Proposition 4 we observe a more general expression for $G(t)$; we construct the approximation of the solution of problem (1), (2) and estimate the distance from solution (4).

ESTIMATION OF THE COEFFICIENTS c_i

In order to obtain the error's estimation which has been named before, let us observe a relation between the coefficients c_i from (4) and a_i from (3).

PROPOSITION 1. *Let us suppose that the following inequalities for coefficients* a_i *hold:*

(7) $$r^{-i}(-1)^{i-1}\binom{r}{i} \le (-1)^{i-1} a_i \le (-1)^{i-1} s^{-i}\binom{s}{i}$$

for two real numbers r,s, *such that* $0 < s < r < 1$. *Then for coefficients* c_i *we have:*

(8) $r^{-1}(-1)^{i-1}\binom{r/2}{i} \le (-1)^{i-1}c_i \le (-1)^{i-1}\binom{s/2}{i}s^{-i}$.

P r o o f. The function $(-1)^{i-1}\frac{1}{x^i}\binom{x}{i} =$

$= \frac{(1-x)(2-x)\dots(i-1-x)}{i!\,x^{i-1}}$ is monotonically decreasing for each

$i \ge 2$ in the interval $0 < x < 1$, hence relation (7) makes sense. For $i=1$ (8) holds; this follows at once from (5). Suppose that (8) holds for $i = n-1$, $n \ge 2$, then

$$(-1)^{n-1}c_n = \frac{1}{2}\left[(-1)^{n-1}a_n + \sum_{j=1}^{n-1}(-1)^{n-j-1}c_{n-j}(-1)^{j-1}c_j\right] \ge$$

$$\ge \frac{1}{2}\left[(-1)^{n-1}r^{-n}\binom{r}{n} + (-1)^n r^{-n}\sum_{j=o}^{n}\binom{r/2}{n-j}\binom{r/2}{j} + \frac{2(-1)^{n-1}}{r^n}\binom{r/2}{n}\right]$$

$$\ge (-1)^{n-1}r^{-n}\binom{r/2}{n}.$$

In the same manner one proves the other inequality in (8).

PROPOSITION 2. *Let us suppose that for the coefficients* a_i *from (3) the following inequalities hold :* $|a_i| \le b_i$, $i=1,2,\dots$, $b_i \ge 0$. *Also, we suppose that there exists* $r > 0$ *such that for each* z, $|z| \le r$ *the following conditions are satisfied:*

a) the series $\sum_{i=1}^{\infty} b_i z^i = K(z)$ *converges ;*

b) $\tilde{G}(z) = \sum_{i=1}^{\infty} a_i z^i \ne -1$

Then $c_i \le M/r^i$, $M = \sqrt{1 + K(r)}$

P r o o f. It follows from the fact that the function $1 + \tilde{G}(z)$ is regular, different from zero in the circle $\{z; |z| \le r\}$ and from Cauchy inequalities.

COROLLARY. *Provided that* $|a_i| \leq C\gamma^i$ *for* $i \geq 1$ *and that there exists* $r > 0$ *such that* $\gamma < 1/r$ *and* $C < (1-\gamma r)/\gamma r$, *then* $|c_i| \leq \sqrt{2}/r^i$.

P r o o f. Let us show that the conditions of Proposition 2 are satisfied for $|z| \leq r$:

a) $|K(z)| = |\sum_{i=1}^{\infty} b_i z^i| \leq C\gamma r \sum_{i=o}^{\infty} (\gamma r)^i \leq C\gamma r/(1-\gamma r)$;

b) $|\tilde{G}(z)| \leq K(|z|) \leq C\gamma r/(1-\gamma r) < 1$, $\tilde{G}(0) = 0$ and it follows

$\tilde{G}(z) \neq -1$;

c) $M = \sqrt{1+K(|z|)} \leq \sqrt{2}$.

APPROXIMATION OF THE SOLUTION OF EQUATION (1) AND THE MEASURE OF APPROXIMATION

It is known from |5| that the solution of problem (1),(2) for G(t) given by (6) and $R(t) \equiv 0$ is:

(9) $$u_\lambda(x) = \frac{1}{2}\ell \ \exp(-|x|s)\exp(-\lambda|x|s^{1-\alpha}) \quad \text{or}$$

(10) $$u_\lambda(x,t) = \frac{1}{2}\ell \begin{cases} (t-|x|)^{-1}\phi(0,-(1-\alpha),-\lambda|x|(t-|x|)^{-(1-\alpha)}), & t > |x| \\ 0, & t \leq |x| \end{cases}$$

where ϕ is Wright's function |6|.

For two elements f and g from C let us denote

$f \leq_T g \Leftrightarrow f(t) \leq g(t)$, $0 \leq t \leq T$.

PROPOSITION 3. *Let us suppose that:*

a) G(t) *is given by relation (3) where* $a_1 = 2\lambda$, $a_2 = \lambda^2$, $\lambda > 0$;

b) The coefficients a_i, $i \geq 1$, *satisfy the conditions of the Corollary to Proposition 2 and*

c) $\alpha > 1/k$, *where* $k > 2$ *such that* $c_i = 0$ *for* $2 \leq i < k$ *and* $c_k \neq 0$.

Then

$$(11) \qquad u(x) = u_\lambda(x)\exp(-|x| \sum_{i=k}^{\infty} c_i \ell^{i\alpha-1})$$

is the solution of equation (1) with conditions (2). For $|x| \leq t \leq T\ (\lambda|x|)^{1/\sigma} + |x| \equiv T_1(|x|)$ *and* $0 \leq t \leq T$, $\sigma = 1-\alpha$, *we have*

$$(12) \qquad |u(x)-u_\lambda(x)| \leq \frac{\nu|x|^{1-1/\sigma}\lambda^{-1/\sigma}T_1^{\delta+1}}{2\delta(\delta+1)}\Big(\frac{T^{2n}}{(2n)!}C(2n+1) + \frac{T^{2n+1}}{(2n+1)!}C(2n+2)\Big)\cdot \sum_{k=o}^{\infty} \frac{(|x|\ \nu T_1^{\delta})^k}{\Gamma(k+2)\Gamma((k+1)\delta)}$$

where

$$(13) \qquad C(k) = \frac{1}{\sigma\pi}\ \Gamma\Big(\frac{k}{\sigma}\Big)\cdot\cos^{-k/\sigma}\frac{\sigma\pi}{2}\ ;\ \delta = k\alpha-1;\ \nu \geq \frac{M}{r^k} \sum_{j\geq o} \frac{T_1^{j\alpha}}{r^j\Gamma(j\alpha+1)} \ .$$

P r o o f. The assumptions of Proposition 3 imply the conditions of Proposition 2 from |5|, hence there exists a solution of form (4) which is a continuous function for $t \geq 0$, $x \neq 0$ having a line of symmetry $x = 0$ and satisfying conditions (2). The coefficients c_i are given by (5). Since $c_2 = \frac{1}{2}(a_2 - (a_1/2)^2) = 0$, it follows that $k \geq 3$. The coefficients a_i satisfy the conditions of the Corollary to Proposition 2 and so $|c_i| \leq \sqrt{2}/r^i$.

The functions $u(x)$ and $u_\lambda(x)$ are continuous, so we can use the inequality of the form $\leq_T$. From (11) follows:

$$|u(x) - u_\lambda(x)| \leq_T |u_\lambda(x)||\exp(-|x| \sum_{i=k}^{\infty} c_i \ell^{i\alpha-1}) - I|$$

Using (10) and |1| we have

$$(14) \qquad u_\lambda(t,x) = \tfrac{1}{2}\ell(\lambda|x|)^{-1/\sigma}|[(\lambda|x|)^{-1/\sigma}(t-|x|)]^{-1}\phi(0,-\sigma, -(\lambda|x|)^{-1/\sigma}(t-|x|)^{-\sigma})| \leq \tfrac{1}{2}\ \ell^2(\lambda|x|)^{-1/\sigma}\Big(\frac{T^{2n}}{(2n)!}C(2n+1) + \frac{T^{2n+1}}{(2n+1)!}C(2n+2)\Big) = \tfrac{1}{2}\ \ell^2 Q(\lambda)$$

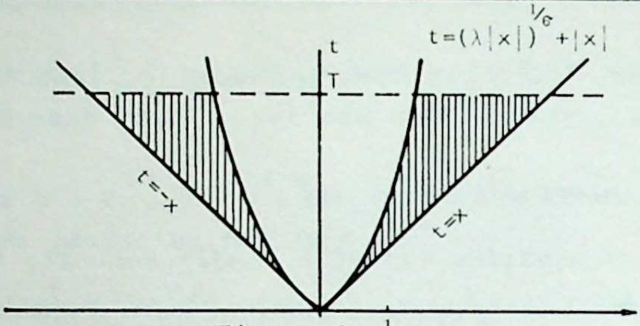

Figure 1

for $|x| \leq t \leq T(\lambda|x|)^{1/\sigma} + |x| = T_1$ or (Fig. 1) $0 \leq (\lambda|x|)^{-1/\sigma} \cdot (t-|x|) \leq T$ and $C(k)$ is given by (13).

Applying the results from |2|, we have

$$|\exp(-|x| \sum_{i=k}^{\infty} c_i \ell^{i\alpha-1}) - I| \leq T\Omega_{\delta,\nu}\ell^{\delta} \quad , \quad \text{where}$$

$$(15) \qquad \Omega_{\delta,\nu} = |x| \nu \Gamma(\delta) \sum_{k=o}^{\infty} \frac{(\nu|x|T_1^{\delta})^k}{\Gamma(k+2)\Gamma((k+1)\delta)}$$

for $0 \leq t \leq T$.

REMARK

This approximation is suitable for $0 \leq T \leq 1$. If $T > 1$, one can use the following estimation for the function ϕ (|4|):

$$(16) \qquad t^{-1}\phi(0,-\sigma,t^{-\sigma}) \leq \frac{1}{\sigma\pi} \cos^{-1/\sigma}(\frac{\sigma\pi}{2})\Gamma(1/\sigma) \equiv K(\sigma) \; .$$

The coefficients a_1 and a_2 are not always supposed to have such a special form as in Proposition 3. We kept them in this form in order to show the influence of the neglecting of addend $R(t)$ in function $G(t)$ on the solution of equation (1), as was done in |3|.

We are going to show only one statement of a more general type:

PROPOSITION 4. *Let us suppose that:*

a) $G(t)$ *is given by (3), where* $a_1 > 0, a_2 > (a_1/2)^2$;

b) *Coefficients* a_i , $i \geq 1$, *satisfy the conditions of the Corollary to Proposition 2;*

c) $1/3 < \alpha < 1/2$.

Then $\tilde{u}_{k-1}(x) = \frac{1}{2}\,\ell \exp(-|x| \sum_{i=0}^{k-1} c_i \ell^{i\alpha-1})$ *is a solution of problem (1), (2) and*

$$(17) \quad |u(x)-\tilde{u}_{k-1}(x)| \leq_T \frac{1}{2}\,\ell^3 \Omega(c_1) \frac{K(1-2\alpha)}{(|x|c_2)^{1/(2\alpha-1)}} \prod_{i=3}^{k-1} (I + \ell^{i\alpha-1}\Omega_{i\alpha-1',|x|c_i}) \cdot \Omega_{\delta,\nu}\ell^{\delta}$$

for $|x| \leq t \leq T(\lambda|x|)^{1/\sigma} + |x| \leq T_1(|x|)$ *and* $0 \leq t \leq T$.

(The notations are as in Proposition 3.)

P r o o f. Let us observe that

$$|\exp(-|x|c_2)\ell^{2\alpha-1}| = |t^{-1}\phi(0,-(1-2\alpha),-|x|c_2 t^{2\alpha-1})| \leq \frac{K(1-2\alpha)\,\ell}{(|x|c_2)^{1/2\alpha-1}}$$

and from |4|

$$|\ell\exp(-|x|c_i\ell^{i\alpha-1})| = |\ell + \ell \sum_{k=1}^{\infty} (-|x|c_i\ell^{i\alpha-1})^k|$$

$$\leq_T \ell + \Omega_{i\alpha-1,|x|c_i}\ell^{i\alpha} .$$

NUMERICAL EXAMPLE

Let us suppose that the coefficients a_i satisfy the conditions of Proposition 3 and the inequalities $|a_i| \leq C\gamma^i$, $i \geq 1$; (for $\lambda = \gamma < 1/r$ and $C < (1/\gamma r)-1$ the coefficients c_i from the Corollary to Proposition 3 satisfy $|c_i| < \sqrt{2}/r^i$). Then the measure of approximation, can be expressed for $|x|=1$:

	λ	$1/r$	$\lvert u(x) - u_\lambda(x) \rvert \leq$		
			$T_1 = 1$ $(T=10^{-5})$	$T_1=2$	$T_1=10$
	10^{-2}	10^{-1}	$3.5611 \cdot 10^{-7}$	$1.5570 \cdot 10^{1}$	$2.7395 \cdot 10^{2}$
	10^{-3}	10^{-2}	$3.4500 \cdot 10^{-8}$	$1.3772 \cdot 10^{0}$	$1.5594 \cdot 10^{1}$
$\alpha=1/2$	10^{-4}	10^{-3}	$3.4291 \cdot 10^{-9}$	$1.3639 \cdot 10^{-1}$	$1.5283 \cdot 10^{0}$
$k=3$	10^{-5}	10^{-4}	$3.2744 \cdot 10^{-10}$	$1.3654 \cdot 10^{-2}$	$1.5254 \cdot 10^{-1}$
	10^{-6}	10^{-5}	$3.2744 \cdot 10^{-11}$	$1.3654 \cdot 10^{-3}$	$1.5251 \cdot 10^{-2}$
	10^{-7}	10^{-6}	$3.2744 \cdot 10^{-12}$	$1.3654 \cdot 10^{-4}$	$1.5251 \cdot 10^{-3}$
	10^{-2}	10^{-1}	$3.6529 \cdot 10^{-3}$	$7.5038 \cdot 10^{0}$	
$\alpha=1/2$	10^{-3}	10^{-2}	$2.0021 \cdot 10^{-4}$	$6.1189 \cdot 10^{-1}$	$2.6787 \cdot 10^{2}$
$k=5$	10^{-4}	10^{-3}	$1.9884 \cdot 10^{-5}$	$6.1168 \cdot 10^{-2}$	$2.5642 \cdot 10^{1}$
	10^{-5}	10^{-4}	$1.9870 \cdot 10^{-6}$	$6.1091 \cdot 10^{-3}$	$2.5544 \cdot 10^{0}$
	10^{-6}	10^{-5}	$1.9870 \cdot 10^{-7}$	$6.1091 \cdot 10^{-4}$	$2.5544 \cdot 10^{-1}$
	10^{-7}	10^{-6}	$1.9870 \cdot 10^{-8}$	$6.1091 \cdot 10^{-5}$	$2.5544 \cdot 10^{-2}$

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REZIME

JEDNAČINA OSCILACIJE ŽILAVOELASTIČNOG ŠTAPA (II)

Ovaj rad je nastavak rada |5|. U radu je ocenjena razlika rešenja jednačine

$$\partial_t^2 u(t,x) - \partial_x^2 u(t,x) + \int_0^t \partial_t^2 u(t-\tau,x)G(\tau)d\tau = 0$$

sa početnim uslovima

$$u(0,x) = 0 \qquad \partial_t u(0,x) = \delta(x)$$

kada G(t) ima oblik

$$G(t) = 2\lambda t^{\alpha-1}/\Gamma(\alpha) + \lambda t^{2\alpha-1}/\Gamma(2\alpha) + R(t)$$

i rešenja koje se dobija kada se u tom izrazu zanemari R(t) (Tvrdjenje 3)

U tvrdjenju 4 posmatran je izraz G(t) oblika:

$$G(t) = \sum_{i=1}^{\infty} a_i \ell^{i\alpha} \quad ;$$

formirana su aproksimativna rešenja koja takodje zadovoljavaju početni uslov i ocenjena je greška.

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ON A COMMON FIXED POINT IN BANACH AND RANDOM NORMED SPACES

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In |1| the following common fixed point theorem is proved.

THEOREM A *Let* S *and* T *be continuous mappings of a complete metric space* (X,d) *into itself. Then* S *and* T *have a common fixed point in* X *if and only if there exists a continuous mapping* $A : X \to SX \cap TX$, *which commutes with* S *and* T *and satisfies the inequality*

$$d(Ax,Ay) \leq q\, d(Sx,Ty) \quad \textit{for every} \quad x,y \in X,$$

where $0 \leq q < 1$. *Indeed* S,T *and* A *then have a unique common fixed point.*

We shall prove in this note a common fixed point theorem if $(X, \|\ \|)$ is a Banach space, S and T are linear mappings,

$$\|Ax-Ay\| \leq \|Sx-Ty\| \quad \text{for every} \quad x,y \in X,$$

and in iterate A^m $(m \in N)$ is ψ-densifying. Here S,T and A are defined on X and $AX \subseteq SX \cap TX$. If $S = T = Id|X$, from our Theorem follows the result from |2| for nonexpansive mapping A. First, we shall give some definitions |4|. Let $(X, \|\ \|)$ be a Banach space, 2^X the set of all subsets of X, and let $\mathcal{N} \subset 2^X$ be such that $Q \in \mathcal{N}$ implies $\overline{co}\, Q \in \mathcal{N}$. Further, let $(\mathcal{U}, \leq)$ be a partially ordered set. A mapping $\psi : \mathcal{N} \to \mathcal{U}$ is a measure of noncompactness if and only if

$$\psi(\overline{co}\ Q) = \psi(Q) \quad \text{for every} \quad Q \in N.$$

The measure ψ is <u>monotone</u> if $Q_1 \subseteq Q_2$ $(Q_1, Q_2 \in N)$ implies $\psi(Q_1) \leq \psi(Q_2)$, and 2-<u>regular</u> if for every totally bounded set $Q \in N$ the relation $\psi(Q) = 0$ holds. The measure ψ is algebraically semi-additive if for every $Q_1, Q_2 \in N$ the inequality $\psi(Q_1 + Q_2) \leq \psi(Q_1) + \psi(Q_2)$ holds. Let $M \subseteq X$ and $F : M \to X$. The mapping F is ψ-densifying iff $Q \subseteq M$ implies $Q \in N$, $F(Q) \in N$ and:

$$\bar{Q} \text{ is not compact} \Rightarrow \psi(F(Q)) \not\geq \psi(Q) .$$

In the following text we shall suppose that the set U is totally ordered and ψ is monotone, 2-regular and algebraically semi-additive, where N is the set of all bounded subsets of Banach space X. Let $X' = \{\lambda y \mid \lambda \in (0,1),\ y \in AX\}$.

THEOREM 1. *Let* $(X, \| \ \|)$ *be a Banach space,* S *and* T *linear, continuous mappings from* X *into* X. *Let* A *be a continuous mapping from* X *into* $SX \cap TX$ *such that* AX *is bounded and that the following two conditions are satisfied:*

1. $\|Ax - Ay\| \leq \|Sx - Ty\|$ *for every* $x, y \in X$.

2. There exists $m \in N$ *such that* $A^m | X'$ *is* ψ *-densifying. If* A *commutes with* S *and* T *then there exists* $x \in X$ *such that*

$$x = Tx = Sx = Ax.$$

P r o o f. Suppose that $\{r_n\}_{n \in N}$ is a sequence of real numbers from $(0,1)$ such that $\lim\limits_{n \to \infty} r_n = 1$. For every $n \in N$ let $A_n x = r_n Ax$, $x \in X$. Let us show that for every $n \in N$ there exists $x_n \in X$ such that:

$$(1) \qquad x_n = A_n x_n = Sx_n = Tx_n .$$

First, from $AX \subseteq SX \cap TX$ and the fact that S and T are linear it follows that:

$$A_n X \subseteq SX \cap TX.$$

Further, $A_nSx = r_nASx = r_nSAx$ and $SA_nx = S(r_nAx) = r_nSAx$ for every $x \in X$, and so A_n and S are commutative and similarly A_n and T. Since for every $x,y \in X$, $\|A_nx-A_ny\| \le r_n \|Sx-Ty\|$, it follows that all the conditions of Theorem A are satisfied. So for every $n \in N$ there exists $x_n \in X$ such that (1) holds. From (1) we have that $x_n-Ax_n = (r_n-1)Ax_n$ and since AX is bounded and $\lim\limits_{n\to\infty} r_n = 1$ it follows that $\lim\limits_{n\to\infty} (x_n-Ax_n) = 0$. Let us prove that:

$$\lim_{n\to\infty} (x_n - A^mx_n) = 0.$$

First we shall show that for every $k,n \in N$

$$\|A^kx_n-A^{k+1}x_n\| \le \|x_n-Ax_n\| .$$

We use induction in k. For $k=1$ and $n \in N$ we have:

$$\|Ax_n-A^2x_n\| \le \| Sx_n-TAx_n\| = \|Sx_n-ATx_n\| = \|x_n-Ax_n\| .$$

Suppose that for some k and every $n \in N$:

$$\|A^kx_n - A^{k+1}x_n\| \le \| x_n - Ax_n\| .$$

Then:

$$\|A^{k+1}x_n-A^{k+2}x_n\| \le \|S(A^kx_n)-T(A^{k+1}x_n)\| = \|A^k(Sx_n)- - A^{k+1}(Tx_n)\| = \|A^kx_n-A^{k+1}x_n\| \le \|x_n-Ax_n\| .$$

Now:

$$\|x_n-A^mx_n \| \le \sum_{s=o}^{m-1} \|A^sx_n-A^{s+1}x_n\| \le m \|x_n-Ax_n\| \quad \text{for every } n \in N ,$$

and since $\lim\limits_{n\to\infty} (x_n-Ax_n) = 0$, it follows that $\lim\limits_{n\to\infty} (x_n-A^mx_n) = 0$.

Let us prove that there exists a convergent subsequence $\{x_{n_k}\}_{k\in N}$.

Let $y_n = x_n-A^mx_n$ for every $n \in N$. Then:

$$\psi[\{x_n | n \in N\}] \le \psi[\{y_n | n \in N\}] + \psi[\{A^mx_n\, n \in N\}]$$

since the measure ψ is monotone and algebraically semi-additive. Since $\lim_{n\to\infty} y_n = 0$ and the measure ψ is 2-regular it follows that $\psi[\{y_n | n \in N\}] = 0$. Consequently:

$$\psi[\{x_n | n \in N\}] \leq \psi[\{ A^m x_n | n \in N \}]$$

Hence, as $A^m|X'$ is ψ densifying it follows that the set $\overline{\{x_n | n \in N\}}$ is compact. Suppose that $\lim_{k\to\infty} x_{n_k} = y^*$. Then $y^* = \lim_{k\to\infty} x_{n_k} = A(\lim_{k\to\infty} x_{n_k}) = S(\lim_{k\to\infty} x_{n_k}) = T(\lim_{k\to\infty} x_{n_k})$, and so y^* is a common fixed point for the mappings A,S and T.

A special 2-regular, monotone and algebraically semi-additive measure ψ is Kuratowski's measure of noncompactness α defined on bounded subsets $A \subseteq X$ by

$$\alpha(A) = \inf\{\varepsilon | \varepsilon > 0, \text{ there exists a finite cover } \mathcal{A} \text{ of the set } A \text{ such that } \operatorname{diam}(B) \leq \varepsilon, \text{ for every } B \in \mathcal{A}\}.$$

From Theorem 1 we obtain the following Corollary.

COROLLARY 1. *Let* X,T,S *and* A *be as in Theorem 1, where* $\alpha = \psi$. *Then there exists a common fixed point for* A,S *and* T.

If $A^m X$ is relatively compact, then A^m is α densifying. This special case can be generalized to random normed spaces (X, F, t) with continuous T-norm t.

A triplet (X,F,t) is a Menger space iff X is an arbitrary set, $F: X \times X \to \Delta$, where Δ denotes the set of all distribution functions F, and t is a T-norm so that the following conditions are satisfied (we write $F(p,q) = F_{p,q}$ for every $p,q \in X$):

1. $F_{p,q}(x) = 1$ for every $x \in R^+$ iff $p = q$.
2. $F_{p,q}(0) = 0$ for every $p,q \in X$.
3. $F_{p,q} = F_{q,p}$ for every $p,q \in X$.
4. $F_{p,r}(x+y) \geq t(F_{p,q}(x), F_{q,r}(y))$ for every $p,q,r \in X$ and every $x,y \geq 0$.

The (ε,λ) topology is introduced by the (ε,λ) neighbourhoods of $v \in X$:

$$U_v(\varepsilon,\lambda) = \{u | F_{u,v}(\varepsilon) > 1-\lambda\}, \ \varepsilon > 0, \ \lambda \in (0,1) \ .$$

In |3| the following Theorem is proved.

THEOREM B. *Let* (X,F,t) *be a complete Menger space with continuous* T-*norm* t , *and let* S *and* T *be continuous mappings* X *into* X. *Then* S *and* T *have a common fixed point in* X *if and only if there exists a continuous mapping* A *of* X *into* $SX \cap TX$ *which commutes with* S *and* T *and satisfies the following two conditions:*

(i) *For every* $x,y \in X$

$F_{Ax,Ay}(\varepsilon) \geq F_{Sx,Ty}(\frac{\varepsilon}{q})$ *for every* $\varepsilon > 0$, *where* $q \in (0,1)$.

(ii) *There exists* $x_o \in X$ *such that* $\sup_{\varepsilon} \inf_{n \in N} F_{Ax_n,Ax_o}(\varepsilon) = 1$.

where $\{x_n\}_{n \in N}$ *is such that* $Ax_{2n-2} = Sx_{2n-1}, Ax_{2n-1} = Tx_{2n}$ *for every* $n \in N$.

Indeed S, T *and* A *then have a unique common fixed point.*

Let S be a real or complex linear space and Δ^+ be the set of all distribution functions F such that $F(0) = 0$. A random normed space is an ordered triple (S,F,t), where t is a T-norm stronger than $T_m : T_m(u,v) = \max\{u+v-1,0\}$ and F is a mapping of S into Δ^+ so that the following conditions are satisfied (we shall denote $F(p)$ by F_p):

1. $F_p = H \Leftrightarrow p = 0$ (0 is the neutral element in S).
2. $F_{\lambda p}(x) = F_p(\frac{x}{|\lambda|})$, for every $p \in S$, $x \in R$ and $\lambda \in K \setminus \{0\}$ where K is the scalar field.
3. $F_{p+q}(x+y) \geq t(F_p(x), F_q(y))$, for every $p,q \in S$ and every $x,y \in R$.

The (ε,λ)-topology in (S,F,t) is introduced by the family of (ε,λ)-neighbourhoods of $v \in S$: $U_v(\varepsilon,\lambda) = \{u | u \in S, F_{u-v}(\varepsilon) > 1-\lambda\}$

where $\varepsilon > 0$ and $\lambda \in (0,1)$ and if T-norm t is continuous then S is, in the (ε,λ)-topology, a Hausdorff linear topological space. Every random normed space is a Menger space if we take $F_{u,v} = F_{u-v}$, for every $u,v \in S$.

From Theoremm B it is easy to obtain the following Corollary in which (X,F,t) is a random normed space.

COROLLARY 2. *Let* (X,F,t) *be a complete random normed space with continuous* T-*norm* t *and let* S *and* T *be continuous mappings from* X *into* X. *If* A *is a continuous mapping from* X *into* $SX \cap TX$ *which commutes with* S *and* T, *if* AX *is bounded in* (ε,λ) *topology and if*

$$F_{Ax-Ay}(\varepsilon) \geq F_{Sx-Ty}\left(\frac{\varepsilon}{q}\right) \qquad \textit{for every } \varepsilon > 0$$

and every $x,y \in X$, *where* $q \in (0,1)$ *then* S,T *and* A *have a unique common fixed point.*

Using the similar idea as in the proof of Theorem 1 we shall prove the following common fixed point theorem.

THEOREM 2. *Let* (X,F,t) *be a complete random normed space with continuous* T-*norm* t, S *and* T *be linear continuous mappings from* X *into* X. *Further let* A *be a continuous mapping which commutes with* S *and* T *such that* $AX \subseteq SX \cap TX$, AX *is bounded in the* (ε,λ) *-topology and* $A^m X$ *is relatively compact. If for every* $x,y \in X$ *and every* $\varepsilon > 0$:

$$F_{Ax-Ay}(\varepsilon) \geq F_{Sx-Ty}(\varepsilon)$$

then there exists $x \in X$ *such that* $x = Tx = Sx = Ax$.

P r o o f. As in the proof of Theorem 1, using Corollary 2, we conclude that, for every $n \in N$, there exists $x_n \in X$ such that

$$x_n = A_n x_n = Sx_n = Tx_n$$

and $\lim_{n\to\infty}(x_n - Ax_n) = 0$. Let us prove that $\lim_{n\to\infty}(x_n - A^m x_n) = 0$. Similarly as in Theorem 1 it follows that for every $k \in N$, every $n \in N$ and every $\varepsilon > 0$:

$$F_{A^k x_n - A^{k+1} x_n}(\varepsilon) \geq F_{x_n - Ax_n}(\varepsilon) .$$

Further from the definition of a random normed space it follows:

$$F_{x_n - A^m x_n}(\varepsilon) \geq t(F_{x_n - Ax_n}(\tfrac{\varepsilon}{2}), F_{Ax_n - A^m x_n}(\tfrac{\varepsilon}{2})) \geq$$

$$\geq t(F_{x_n - Ax_n}(\tfrac{\varepsilon}{2}), t(F_{Ax_n - A^2 x_n}(\tfrac{\varepsilon}{4}), F_{A^2 x_n - A^m x_n}(\tfrac{\varepsilon}{4}))) \geq$$

$$\geq t(F_{x_n - Ax_n}(\tfrac{\varepsilon}{2}), t(F_{x_n - Ax_n}(\tfrac{\varepsilon}{4}), t(F_{x_n - Ax_n}(\tfrac{\varepsilon}{2^3}), \ldots$$

$$\ldots, F_{x_n - Ax_n}(\tfrac{\varepsilon}{2^{m-1}})))).$$

Since $t(1,1) = 1$, t is continuous and

$$\lim_{n\to\infty} F_{x_n - Ax_n}(\tfrac{\varepsilon}{2^s}) = 1, \; s = 1,2,\ldots,m-1,$$

it follows that $\lim_{n\to\infty} F_{x_n - A^m x_n}(\varepsilon) = 1$ for every $\varepsilon > 0$, which means that $\lim_{n\to\infty} x_n - A^m x_n = 0$. The rest of the proof is similar to that of Theorem 1.

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REZIME

O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI U BANAHOVIM I SLUČAJNIM NORMIRANIM PROSTORIMA

U ovom radu su dokazane sledeće dve teoreme.

TEOREMA 1. Neka je $(X, \|\ \|)$ *Banachov prostor,* S *i* T *linearna preslikavanja iz* X *u* X. *Neka je* A *neprekidno preslikavanje* X *u* $SX \cap TX$ *tako da je* AX *ograničen skup i da su zadovoljeni sledeći uslovi,* gde je $X' = \{\lambda y \mid \lambda \in (0,1),\ y \in AX\}$.

1. $\|Ax-Ay\| \leq \|Sx-Ty\|$ *za svako* $x,y \in X$.
2. *Postoji* $m \in N$ *tako da je* $A^m \mid X'$ ψ *kondenzujuće preslikavanje, gde je mera nekompaktnosti* ψ *monotona, 2-regularna i algebarski semiaditivna.*

Ako preslikavanje A *komutira sa preslikavanjima* S *i* T *tada postoji* $x \in X$ *tako da je* $x = Tx = Sx = Ax$.

TEOREMA 2. *Neka je* (X,F,t) *kompletan slučajan normirani prostor sa neprekidnom* T*-normom* t, S *i* T *linearna neprekidna preslikavanja iz* X *u* X. *Dalje, neka je* A *neprekidno preslikavanje, koje komutira sa* S *i* T *tako da je* $AX \subseteq SX \cap TX$, AX *je ograničeno u* (ε,λ)*-topologiji i* $A^m X$ *je relativno kompaktan skup. Ako za svako* $x,y \in X$ *i svako* $\varepsilon > 0$ *važi nejednakost* $F_{Ax-Ay}(\varepsilon) \geq F_{Sx-Ty}(\varepsilon)$ *tada postoji* $x \in X$ *tako da je* $x = Tx = Sx = Ax$.

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A GENERALIZATION OF KAKUTANI'S FIXED POINT THEOREM IN PARANORMED SPACES

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ABSTRACT

In this paper the following theorem is proved:

Suppose that $(X, \|\ \|_1^*)$ *and* $(Y, \|\ \|_2^*)$ *are paranormed spaces,* Z *is a compact and convex subset of* X, K *is a compact and convex subset of* Y, f *is an upper semicontinuous mapping from* Z *into the set* R(K) *of all closed and convex subsets* $M \subset K$, $M \neq \emptyset$ *and* $p : K \to Z$ *is a continuous mapping. If* f(Z) *and* p(co f(Z)) *satisfy the Zima condition then there exists* $x_o \in Z$ *so that* $x_o \in p(f(x_o))$.

From this Theorem two Corollaries are obtained.

1. Let E be a linear space over the real or complex number field. The function $\|\ \|^* : E \to [0, \infty)$ will be called paranorm if and only if :

1. $\|x\|^* = 0 \Leftrightarrow x = 0$.
2. $\|-x\|^* = \|x\|^*$, for every $x \in E$.
3. $\|x+y\|^* \leq \|x\|^* + \|y\|^*$, for every $x, y \in E$.
4. If $\|x_n - x_o\|^* \to 0$, $\lambda_n \to \lambda$ then $\|\lambda_n x_n - \lambda_o x_o\|^* \to 0$.

The function $\rho : E \times E \to [0, \infty)$ defined by $\rho(x,y) = \|x-y\|^*$ is the distance function on E. If (E, ρ) is the complete metric space then it is a Fréchet space. Further $(E, \|\ \|^*)$ is a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family $\{U_\varepsilon\}_{\varepsilon>0}$ where:

$$U_\varepsilon = \{x | x \in E, \ ||x||^* < \varepsilon\} .$$

In |8| the following Theorem is proved, where $(E, || \ ||^*)$ is paranormed space.

THEOREM 1. *Let* K *be a bounded, closed and convex subset of* E *and* $A : K \to K$ *be a completely continuous operator on* K *. If there exists a number* $C > 0$ *such that:*

$||\lambda x||^* \le C\lambda \ ||x||^*$ *, for every* $0 \le \lambda \le 1$ *and every* $x \in A(K) - A(K)$ *then there exists an element* $p \in K$ *with* $A(p) = p$.

The above fixed point theorem can be applied |8| in the proof of the existence of a solution of the infinite system of integral equations:

$$x_i = \int_0^t f_i(s, A_{i_1}(x_1), A_{i_2}(x_2), \ldots, A_{i_{n_i}}(x_{n_i}))ds, \ i=1,2,\ldots .$$

In |3| and |4| some fixed point theorems for multivalued mappings in paranormed spaces are proved.

DEFINITION 1. *Let* $(E, || \ ||^*)$ *be a paranormed space and* K *be a nonempty subset of* E. *If there exists* $C(K) > 0$ *such that:*

$||\lambda x||^* \le C(K)\lambda \ ||x||^*$, *for every* $0 \le \lambda \le 1$ *and every* $x \in K-K$

we say that K *satisfies the Zima condition.*

In this paper we shall prove a generalization of Kakutani's fixed point theorem in paranormed space which is similar to the Lemma from |5| .

2. Let X,Y be topological spaces. We shall denote by 2^Y the family of all nonempty subsets of Y. Let $f : X \to 2^Y$. The mapping f is called upper semicontinuous if for each open subset G of Y, the set:

$$\{x | x \in X, \ f(x) \subset G\}$$

is open in X. If $K \subset Y$ and Y is a topological vector space we shall denote by $\mathcal{R}(K)$ the family of all nonempty, closed and convex subsets of K.

Now, we shall prove the following fixed point theorem .

THEOREM 2. *Suppose that* $(X, \| \|_1^*)$ *and* $(Y, \| \|_2^*)$ *are paranormed space , Z is a compact and convex subset of X, K is a compact and convex subset of Y , f is an upper semicontinuous mapping from Z into* $\mathcal{R}(K)$ *and* $p : K \to Z$ *is a continuous mapping. If* $f(Z)$ *and* $p(\mathrm{co}\, f(Z))$ *satisfy the Zima condition then there exists* $x_o \in Z$ *such that* $x_o \in p(f(x_o))$.

P r o o f. In the proof we shall use some ideas from |6|. Since Z is compact for every $\varepsilon > 0$ there exists a finite ε-net of the set Z, $\{x_{\varepsilon,1}, x_{\varepsilon,2}, \ldots, x_{\varepsilon,n(\varepsilon)}\}$. As in |6|, let the family $\{\omega_{\varepsilon,i}(x)\}_{i=1}^{n(\varepsilon)}$ be defined by:

$$\omega_{\varepsilon,i}(x) = \frac{g_{\varepsilon,i}(x)}{\sum_{j=1}^{n(\varepsilon)} g_{\varepsilon,j}(x)} \quad (i=1,2,\ldots,n(\varepsilon)),\ x \in Z$$

where $g_{\varepsilon,i}(x) = \max\{\varepsilon - \|x - x_{\varepsilon,i}\|_1^*, 0\}$ $(i=1,2,\ldots,n(\varepsilon))$, $x \in Z$.

Further, let $y_{\varepsilon,i} \in f(x_{\varepsilon,i})$ $(i=1,2,\ldots,n(\varepsilon))$ and, as in |6|:

$$f_\varepsilon(x) = \sum_{i=1}^{n(\varepsilon)} \omega_{\varepsilon,i}(x) y_{\varepsilon,i}, \qquad x \in Z .$$

Since the set K is convex it follows that the mapping f_ε is a continuous mapping from Z into K. Indeed $f_\varepsilon(Z) \subseteq \mathrm{co}\, f(Z)$. Further $p \cdot f_\varepsilon : Z \to Z$ is a continuous mapping and since $p \cdot f_\varepsilon(Z)) \subseteq \subseteq p(\mathrm{co}\, f(Z))$ it follows that the mapping $h_\varepsilon = p \cdot f_\varepsilon$ satisfies all the conditions of Theorem 1. So for every $\varepsilon > 0$ there exists $x_\varepsilon \in Z$ such that $x_\varepsilon = h_\varepsilon(x_\varepsilon)$ and so:

$$x_\varepsilon = p \cdot f_\varepsilon(x_\varepsilon) \tag{1}$$

Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$ and let $u_n = f_{\varepsilon_n}(x_{\varepsilon_n})$ $(n \in N)$. Since $u_n \in \in K$ $(n \in N)$ and the set K is compact there exists a convergent

subsequence $\{u_{n_k}\}_{k\in N}$ of the sequence $\{u_n\}_{n\in N}$ and let $u = \lim_{k\to\infty} u_{n_k}$. If $x = p(u)$ we shall prove that $u \in f(x)$. Since $u_{n_k} = f_{\varepsilon_{n_k}}(x_{\varepsilon_{n_k}})$, for every $k \in N$ from (1) we obtain that:

$$(2) \qquad p(u_{n_k}) = p \cdot f_{\varepsilon_{n_k}}(x_{\varepsilon_{n_k}}) = x_{\varepsilon_{n_k}}, \ k \in N.$$

The mapping p is continuous and so from (2) we obtain that:

$$\lim_{k\to\infty} p(u_{n_k}) = p\,(\lim_{k\to\infty} u_{n_k}) = p(u) = \lim_{k\to\infty} x_{\varepsilon_{n_k}} = x\ .$$

Let δ be an arbitrary positive number. We shall prove that $u \in f(x) + \bar{U}_\delta$ which implies that $u \in f(x)$. Let:

$$G_\delta = f(x) + U_{\frac{\delta}{C(f(Z))}} \ .$$

Since the mapping f is upper semicontinuous there exists $\xi > 0$ so that $f(V_\xi(x)) \subseteq G_\delta$, where $V_\xi(x) = \{z \mid z \in Z, \|z-x\|_1^* < \xi\}$. As in |6| it follows from $\lim_{k\to\infty} \varepsilon_{n_k} = 0$ and $\lim_{k\to\infty} x_{\varepsilon_{n_k}} = x$ that there exists a natural number n_o such that for $k \geq n_o$ we have that $x_{\varepsilon_{n_k},i} \in V_\xi(x)$, for every $i \in \{1,2,\ldots,n(\varepsilon_{n_k})\}$ and the following implication:

$$(3) \qquad \omega_{\varepsilon_{n_k},i}(x_{\varepsilon_{n_k},i}) > 0 \Rightarrow y_{\varepsilon_{n_k},i} \in f(x_{\varepsilon_{n_k},i}) \subset f(V_\xi(x)) \subset G_\delta \ .$$

Since the set f(Z) satisfies the Zima condition it follows easily that:

$$co(U_{\frac{\delta}{C(f(Z))}} \cap (f(Z) - f(Z))) \subseteq U_\delta \ .$$

Since $u_{n_k} = f_{\varepsilon_{n_k}}(x_{\varepsilon_{n_k}})$, for every $k \in N$, from the definition of the mapping $f_{\varepsilon_{n_k}}$ it follows that:

(4) $$u_{n_k} = \sum_{i=1}^{n(\varepsilon)} \omega_{\varepsilon_{n_k},i}(x_{\varepsilon_{n_k},i})y_{\varepsilon_{n_k}},$$

Let us suppose now that $k \geq n_o$. From (3) and (4) we obtain that:

$$u_{n_k} = \sum_{i:\omega_{\varepsilon_{n_k},i}(x_{\varepsilon_{n_k},i})>0} \omega_{\varepsilon_{n_k},i}(x_{\varepsilon_{n_k},i})y_{\varepsilon_{n_k},i}, \quad \text{for every } k \geq n_o .$$

and so:

$$u_{n_k} \in f(x) + \mathrm{co}(U_{\frac{\delta}{C(f(Z))}}(f(Z)-f(Z))) \subseteq f(x) + U_\delta .$$

since $G_\delta = f(x) + U_{\frac{\delta}{C(f(Z))}}$ and the set $f(x)$ is convex.

From the relation $u_{n_k} \in f(x) + U_\delta$, for every $k \geq n_o$ it follows that $u \in f(x) + \overline{U}_\delta$ which completes the proof.

If $X = Y$, $K = Z$, $p = Id|Z$ and X is a normed space from Theorem 2 we obtain the fixed point theorem from |6|.

3. Now, we shall prove a Corollary from Theorem 2. First, we shall give the definition of the generalized contraction |7|.

DEFINITION 2. *Let* (X,d) *be a metric space and* $T:X \to X$. *The mapping* $T : X \to X$ *is a generalized contraction if and only if:*

$d(Tx,Ty) \leq L(r,s)\, d(x,y)$, *for every* $x,y \in X$ *such that* $r \leq d(x,y) \leq s$, *where the function* L *is defined for every* $(r,s) \in \in (0,\infty)$ *such that* $r \leq s$ *and* $L(r,s) < 1$.

If X is complete, then a generalized contraction $T:X \to X$ has one and only fixed point x.

COROLLARY 1. *Let* $(X, \| \ \|_1^*)$ *and* $(Y, \| \ \|_2^*)$ *be paranormed spaces* , X *be complete* ,Z *be a compact and convex subset of* X, K *be a compact and convex subset of* Y, f *be an upper semi-continuous mapping from* Z *into* R(K), $p : K \to K_1 (K_1 \subseteq X)$ *be a continuous mapping and* $T : Z \to K_2 (K_2 \subseteq X)$ *be a generalized contraction so that the following conditions are satisfied:*

(i) *The sets* $T(Z) + p(\overline{co}\ f(Z))$ *and* $f(Z)$ *satisfy the Zima condition, and* $TZ + p(\overline{co}\ f(Z)) \subseteq Z$.

(ii) *The set* $(Id-T)^{-1}\ p(\overline{co}\ f(Z))$ *is bounded.*

Then there exists $x \in Z$ *such that* $x \in Tx + p(f(x))$.

P r o o f. Since $T(Z) + p(\overline{co}\ f(Z)) \subseteq Z$ and T is a generalized contraction for every $y \in p(\overline{co}\ f(Z))$ there exists $Ry \in Z$ such that $Ry = TRy + y$. Let us prove that the mapping $R : p(\overline{co} f(Z)) \to Z$ is continuous. Suppose that $\{x_n\}_{n\in N} \subseteq p(\overline{co}\ f(Z))$ and $\lim_{n\to\infty} x_n = x$. If, on the contrary, the mapping R is not continuous then there exists $\varepsilon > 0$ and a sequence $\{n_k\}_{k\in N} \subseteq N$ such that $\|Rx_{n_k} - Rx\|_1^* \geq \varepsilon$, for every $k \in N$ ($n_k \geq k$, for every $k \in N$). Since the set $(Id-T)^{-1} p(\overline{co}\ f(Z))$ is bounded there exists $M > 0$ so that $\| Ry \|_1^* \leq M$, for every $y \in p(\overline{co}\ f(Z))$. As in |2| it follows that:

$$\|Rx_{n_k} - Rx\|_1^* \leq L(\varepsilon, 2M)\ \|Rx_{n_k} - Rx\|_1^* + \|x_{n_k} - x\|_1^* .$$

Further since $\{\|Rx_{n_k} - Rx\|_1^*\}_{k\in N} \subseteq [\varepsilon, 2M]$, there exists a subsequence $\{x_{n_{k(r)}}\}_{r\in N}$ such that:

$$m = \lim_{r\to\infty} \|Rx_{n_{k(r)}} - Rx\|_1^*$$

and so $m \leq L(\varepsilon, 2M)m < m$ which is a contradiction. Let us define the mapping $g : \overline{co}\ f(Z) \to Z$ in the following way $g(x) = Rp(x)$, for every $x \in \overline{co}\ f(Z)$. Then all the conditions of Theorem 2 are satisfied for $\overline{co}\ f(Z)$ instead of K and g instead of p. So there exists $x \in Z$ such that $x \in g(f(x))$. This means that $x = g(u)$, where $u \in f(x)$. From $x = g(u)$ it follows that $x = Rp(u)$ which implies that:

$x = Rp(u) = TRp(u) + p(u) = Tx + p(u)$, $u \in f(x)$ and so:

$$x \in Tx + pf(x) \ .$$

REMARK: If, in Theorem 2, X and Y are complete paranormed spaces, and K and Z satisfy the Zima condition it is enough to suppose that Z and K are closed and convex and that the set $\overline{f(Z)}$ is compact. Namely, in this case the set $\overline{co}\ f(Z)$ is a compact and convex subset of K and also the set $\overline{co}\ p(\overline{co}\ f(Z)) \subseteq Z$. Then we can apply Theorem 2 taking for the set K the set $\overline{co}\ f(Z)$ and for the set Z the set $\overline{co}\ p(\overline{co}\ f(Z))$. Indeed, it remains to show that $f(\overline{co}\ p(\overline{co}\ f(Z))) \subseteq \overline{co}\ f(Z)$ since we have $p(\overline{co}\ f(Z)) \subseteq \subseteq \overline{co}\ p(\overline{co}\ f(Z))$.

From $p(K) \subseteq Z$ it follows that $\overline{co}\ p(\overline{co}\ f(Z)) \subseteq Z$ and so $f(\overline{co}\ p(\overline{co}\ f(Z))) \subseteq f(Z) \subseteq \overline{co}\ f(Z)$. Similarly as in |1| we shall prove the following Corollary from Theorem 2.

COROLLARY 2. *Suppose that* $(X, \|\ \|_1^*)$ *and* $(Y, \|\ \|_2^*)$ *are complete paranormed spaces,* Z *is a closed and convex subset of* X, K *is a closed and convex subset of* Y, K *and* Z *satisfy the Zima conditions,* f *is an upper semicontinuous mapping from* Z *into* $R(K)$, $p : K \to Z$ *is a continuous mapping and the following conditions are satisfied:*

(i) *There exists* $C \subseteq Z$ *such that* $C \subseteq \overline{co}\ p(\overline{co}\ f(C))$.

(ii) *For every* $Q \subseteq Z$ *such that:* $\overline{co}\ Q = Q$ *we have the following implication:*

$$\overline{co}\ p(\overline{co}\ f(Q)) = Q \Rightarrow Q \text{ is compact.}$$

Then $\mathrm{Fix}(p \cdot f) \neq \emptyset$.

P r o o f. Let the family F be defined in the following way: $F = \{Q, Q \subseteq Z,\ C \subseteq Q,\ Q$ is closed and convex and $p(\overline{co}\ f(Q)) \subseteq Q\}$. Since Z is closed and convex and $p(\overline{co}\ f(Z)) \subseteq \subseteq Z$ it follows that $Z \in F$ and so $F \neq \emptyset$. First, we shall prove that:

$$Q \in F \Rightarrow \overline{co}\ p(\overline{co}\ f(Q)) \in F$$

Since the set $\overline{co}p(\overline{co}\ f(Q))$ is closed and convex it remains to show that $C\subseteq\overline{co}\ p(\overline{co}\ f(Q))$ and that:

(5) $\quad p(\overline{co}\ f(\overline{co}\ p(\overline{co}\ f(Q))))\subseteq\overline{co}\ p(\overline{co}\ f(Q)).$

From $Q\in F$ it follows that $C\subseteq Q$ and so $f(C)\subseteq f(Q)$. From this we have the following implications:

$$\overline{co}\ f(Q)\subseteq\overline{co}\ f(C)\Rightarrow p(\overline{co}f(Q))\subseteq p(\overline{co}f(C))\Rightarrow\overline{co}\ p(\overline{co}f(Q))\subseteq \subseteq\overline{co}\ p(\overline{co}\ f(C))$$

and since $\overline{co}\ p(\overline{co}\ f(C))\subseteq C$ we conclude that $\overline{co}\ p(\overline{co}\ f(Q))\subseteq C$.

Let us prove relation (5). We have the following implications:

$$\overline{co}\ p(\overline{co}\ f(Q))\subseteq Q\Rightarrow f(\overline{co}\ p(\overline{co}\ f(Q)))\subseteq f(Q)\Rightarrow$$
$$\Rightarrow\overline{co}\ f(\overline{co}\ f(\overline{co}\ p(\overline{co}\ f(Q)))\subseteq\overline{co}\ f(Q)\Rightarrow$$
$$\Rightarrow p(\overline{co}\ f(\overline{co}\ p(\overline{co}\ f(Q))))\subseteq p(\overline{co}\ f(Q))\subseteq\overline{co}\ p(\overline{co}\ f(Q))$$

and so from $Q\in F$ it follows that $\overline{co}\ p(\overline{co}\ f(Q))\in F$. Let us denote by K_o the set $\bigcap_{Q\in F} Q$. Since $Q\in F$ implies that $C\subseteq Q$, we have that $C\subseteq\bigcap_{Q\in F} Q=K_o$ and so K_o is a nonempty, closed and convex subset of Z. From $p(\overline{co}\ f(Q))\subseteq Q$ for every $Q\in F$ it follows that

$$\bigcap_{Q\in F} p(\overline{co}\ f(Q))\subseteq\bigcap_{Q\in F} Q$$

and so $p(\bigcap_{Q\in F}\overline{co}\ f(Q))\subseteq\bigcap_{Q\in F} Q$. Since $\overline{co}\bigcap_{i\in I} A_i\subseteq\bigcap_{i\in I}\overline{co}\ A_i$ where $A_i\in K$, for every $i\in I$ we have that:

$$p(\bigcap_{Q\in F}\overline{co}\ f(Q))\supseteq p(\overline{co}\bigcap_{Q\in F} f(Q))$$

which implies that:

$$p(\overline{co}\ f(\bigcap_{Q\in F} Q))\subseteq p(\overline{co}\bigcap_{Q\in F} f(Q))\subseteq\bigcap_{Q\in F} Q$$

and so it follows that $p(\overline{co}\ f(K_o))\subseteq K_o$. Since K_o is closed, convex, $K_o\supseteq C$ and $p(\overline{co}\ f(K_o))\subseteq K_o$ we conclude that $K_o\in F$ and so $\overline{co}\ p(\overline{co}\ f(K_o))\in F$. Further $K_o=\bigcap_{Q\in F} Q$ and so $K_o\subseteq\overline{co}p(\overline{co}f(K_o))$.

Since, on the other hand, $p(\overline{co}\ f(K_o)) \subseteq K_o$ implies that $\overline{co}p(\overline{co}\ f(K_o)) \subseteq K_o$ we conclude that $K_o = \overline{co}\ p(\overline{co}\ f(K_o))$. From (ii) it follows that the set K_o is a compact subset of Z. Now, we can apply Theorem 2 taking for the set Z the set K_o, for the set K the set $\overline{co}\ f(K_o)$, f is $f|K_o$ and p is $p|\overline{co}\ f(K_o)$. Since the mapping $p|\overline{co}\ f(K_o)$ maps $\overline{co}\ f(K_o)$ into the set K_o all the conditions of Theorem 2 are satisfied and so there exists $x_o \in K_o$ such that $x_o \in p(f(x_o))$.

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REZIME

UOPŠTENJE TEOREME KAKUTANIJA O NEPOKRETNOJ TAČKI U PARANORMIRANIM PROSTORIMA

U ovom radu dokazana je sledeća teorema.

TEOREMA. *Neka su* $(X, \| \|_1^*)$ *i* $(Y, \| \|_2^*)$ *paranormirani prostori,* Z *je kompaktan i konveksan podskup od* X *,* K *je kompaktan i konveksan podskup od* Y *,* f *je od gore poluneprekidno preslikavanje* Z *u* $R(K)$ *i* $p : K \to Z$ *je neprekidno preslikavanje. Ako* $f(Z)$ *i* $p(\text{co}\, f(Z))$ *zadovoljavaju Zimin uslov tada postoji* $x_o \in Z$ *tako da je* $x_o \in p(f(x_o))$.

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A THEOREM ON ALMOST CONTINUOUS SELECTION PROPERTY AND ITS APPLICATIONS

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1. INTRODUCTION

E.Michael and C.Pixley proved the following Theorem which unifies and generalizes some previously known results about the almost continuous selection property.

THEOREM 1. |8| *Let* X *be paracompact,* Y *be a Banach space,* $Z \subseteq X$ *with* $\dim_X Z \leq 0$ *and* $\phi : X \to F(Y)$ *a lower semicontinuous mapping with* $\phi(x)$ *convex for all* $x \in X \setminus Z$. *Then* ϕ *admits a selection.*

In this paper we shall prove that a similar result holds also if X is a normal topological space and Y is a paranormed space.

First, we shall give some notations and definitions. Let: $2^Y = \{S \mid S \subseteq Y, S \neq \emptyset\}$ and $F(Y) = \{S \mid S \in 2^Y$ and S is closed in Y$\}$. A mapping $\phi: X \to 2^Y$ is lower semicontinuous (l.c.s.) if and only if the set $\{x \mid x \in X, \phi(x) \cap V \neq \emptyset\}$ is open in X for every open V in Y. A selection for a mapping $\phi: X \to F(Y)$ is a continuous mapping $f: X \to Y$ such that $f(x) \in \phi(x)$ for all $x \in X$. Finally, if $Z \subseteq X$ then $\dim_X Z \leq 0$ means that $\dim E \leq 0$ for every set $E \subset Z$ which is closed in X.

Let E be a linear space over the real or complex number field. The function $\| \|^* : E \to [0,\infty)$ will be called a paranorm if and only if:

1. $\|x\|^* = 0 \iff x = 0$.
2. $\|-x\|^* = \|x\|^*$, for every $x \in E$.
3. $\|x+y\|^* \leq \|x\|^* + \|y\|^*$, for every $x, y \in E$.
4. If $\|x_n - x_o\|^* \to 0$, $\lambda_n \to \lambda_o$ then $\|\lambda_n x_n - \lambda_o x_o\|^* \to 0$.

Then we say that $(E, \| \|^*)$ is a paranormed space. The space E is also a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family $\{U_\varepsilon\}_{\varepsilon > 0}$ where $U_\varepsilon = \{x | x \in E, \|x\|^* < \varepsilon\}$.

In |9| the following fixed point theorem is proved.

THEOREM 2. *Let* K *be a bounded, closed and convex subset of* E *and* $T : K \to K$ *be a completely continuous operator on* K. *If there exists a number* $C(K) > 0$ *such that:*

(1) $\|\lambda x\|^* \le C(K)\lambda \|x\|^*$, *for every* $0 \le \lambda < 1$ *and* $x \in K-K$

then there exists an element $p \in K$ *such that* $Tp = p$.

In |9| Zima has given an example of the space E and of the set K such that the relation (1) is satisfied.

DEFINITION *Let* $(E, \| \|^*)$ *be a paranormed space and* K *be a nonempty subset of* E. *If there exists* $C(K) > 0$ *such that (1) holds we say that* K *satisfies the Zima condition.*

Some fixed point theorems in paranormed spaces are proved in |3|.

2. AN ALMOST CONTINUOUS SELECTION THEOREM

First, we shall prove the following Lemma.

LEMMA 1. *Let* $(Y, \| \|^*)$ *be a paranormed space,* K *be a compact and convex subset of* Y *which satisfies the Zima condition. Then for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *so that:*

$$co((U_\delta + C) \cap K) \subseteq C + U_\varepsilon$$

for every closed and convex subset C *of* K.

P r o o f: Let $\delta > 0$ be such that $U_\delta + U_\delta \subseteq U_{\frac{\varepsilon}{C^2(K)}}$. Since the set C is closed and K is compact there exists a finite set $F = \{x_1, x_2, \ldots, x_n\} \subseteq C$ so that

$$C \subseteq \bigcup_{i=1}^{n} \{x_i + U_\delta\}.$$

Then $(C+U_\delta)\cap K \subseteq \bigcup_{i=1}^{n} \{(x_i + U_{\frac{\varepsilon}{C^2(K)}})\cap K\}$.

Let $\{\beta_k\}_{k=1}^{n}$ be a partition of the unity subordinated to the open covering $\{(x_i + U_{\frac{\varepsilon}{C^2(K)}})\}_{i=1}^{n}$. Suppose now that $z \in co\{(C+U_\delta)\cap K\}$.

Then there exist $\gamma_i \geq 0$ $(i=1,2,\dots,m)$ and $z_j \in \{(C+U_\delta)\cap K\}$ $(j=1,2,\dots,m)$ so that $z = \sum_{j=1}^{m} \gamma_j z_j$. Further, for every $j \in \{1,2,\dots,m\}$ let:

$$c_j = \sum_{k=1}^{n} \beta_k(z_j)x_k .$$

Then from the fact that C is convex it follows that $c_j \in C$, for every $j \in \{1,2,\dots,m\}$ and so $c = \sum_{j=1}^{m} \gamma_j c_j \in C$. Now, we shall prove that $z - c \in U_\varepsilon$, which implies that $z \in U_\varepsilon + C$. Indeed, since the set K satisfies the Zima condition it follows that:

$$\|z-c\|^* = \|\sum_{j=1}^{m} \gamma_j z_j - \sum_{j=1}^{m} \gamma_j c_j\|^* \leq C(K)\sum_{j=1}^{m} \gamma_j \|z_j - c_j\|^* =$$

$$= C(K)\sum_{j=1}^{m} \gamma_j \|z_j - \sum_{k=1}^{n}\beta_k(z_j)x_k\|^* = C(K)\sum_{j=1}^{m} \gamma_j \|\sum_{k=1}^{n} \beta_k(z_j)(z_j - x_k)\|^* \leq$$

$$\leq C^2(K)\sum_{j=1}^{m} \gamma_j (\sum_{\beta_k(z_j)\neq 0} \beta_k(z_j)\|z_j - x_k\|^* \leq$$

$$\leq C^2(K)\sum_{j=1}^{m} \gamma_j (\sum_{\beta_k(z_j)\neq 0} \beta_k(z_j)\frac{\varepsilon}{C^2(K)}) = \varepsilon$$

which means that $z - c \in U_\varepsilon$.

THEOREM 3. *Let* X *be a normal topological space,* $(Y, \|\ \|^*)$ *be a paranormed space,* $Z \subseteq X$ *so that* $\dim_X Z \leq 0$ *and* $\phi : X \to F(Y)$ *be a lower semicontinuous mapping so that* $\phi(x)$ *is convex, for every* $x \in X \setminus Z$. *If* $\phi(X) \subseteq K$ *where* K *is a compact and convex subset of* Y *and* K *satisfies the Zima condition then* ϕ *has the almost continuous selection property which means that for every* $\varepsilon > 0$ *there exists a continuous mapping* $f : X \to K$ *such that* $f(x) \in U_\varepsilon + \phi(x)$, *for every* $x \in X$.

P r o o f: For every $\varepsilon > 0$ we shall denote the set $U_\varepsilon + \phi(x)$ by $B_\varepsilon(\phi(x))$ for every $x \in X$. Let us prove that there exists $f : X \to Y$ such that $f(x) \in B_\varepsilon(\phi(x))$, for every $x \in X$. From Lemma it

follows that for $\varepsilon > 0$ there exists $\delta > 0$ so that for every $x \in X \setminus Z$:

$$\mathrm{co}((U_\delta + \phi(x)) \cap K) \subseteq U_\varepsilon + \phi(x) = B_\varepsilon(\phi(x)) .$$

Since $\{U_\delta + y\}_{y \in K}$ is an open covering of the set K and K is compact, there exists a finite subset $\{y_1, y_2, \ldots, y_n\} \subseteq K$ such that:

$$K \subseteq \bigcup_{i=1}^{n} \{y_i + U_\delta\} .$$

Further, for every $k=1,2,\ldots,n$ let $U_{y_k} \subseteq X$ be defined in the following way:

$$U_{y_k} = \{x \mid x \in X,\ y_k \in B_\delta(\phi(x))\} .$$

Since the mapping ϕ is lower semicontinuous it follows that every set U_{y_k} is an open subset of X. Since X is a normal topological space there exists an open covering $\{V_{y_k}\}_{k=1}^{n}$ such that $\overline{V}_{y_k} \subset U_{y_k}$ for every $k \in \{1,2,\ldots,n\}$. Further let:

$$F_x = \{y_k \mid y_k \in K,\ x \in \overline{V}_{y_k}\} , \text{ for every } x \in X .$$

From the definition of F_x it follows that for every $x \in X$ $F_x \subset B_\delta(\phi(x))$. Let $S = X \setminus Z$ and for every $s \in S$:

$$G_s = \{x \mid x \in X,\ \mathrm{co}\, F_s \subset B_\varepsilon(\phi(x))\} \setminus \{ \bigcup_{y_k \notin F_s} \overline{V}_{y_k} \} .$$

Every set G_s is nonempty, since $s \in G_s$. Namely:

$$\mathrm{co}\, F_s \subset \mathrm{co}(B_\delta(\phi(s)) \cap K) \subset B_\varepsilon(\phi(s)) .$$

Using Lemma 1 we conclude that for every $s \in S$ the set G_s is open. Further, for every $x \in G_s$ we have that $F_x \subset F_s$ since $x \notin \overline{V}_{y_k}$ if $y_k \notin F_s$. Let $G = \bigcup_{s \in S} G_s$ and $E = X \setminus G$. Since G is open the set E is closed and $\dim E \leq 0$. For the relatively open covering $\{V_{y_k} \cap E\}_{k=1}^{n}$ of E there exists a relatively open, disjoint covering $\{D_{y_k}\}_{k=1}^{n}$ such that $D_{y_k} \subseteq V_{y_k} \cap E$ and let $W_{y_k} = V_{y_k} \cap (D_{y_k} \cup G)$ for every $k=1,2,\ldots,n$. Now, we shall define the mapping $f : X \to K$ in the following way:

$$f(x) = \sum_{k=1}^{n} p_{y_k}(x) y_k , \quad \text{for every } x \in X$$

and $\{p_{y_k}\}_{k=1}^{n}$ is the partition of the unity subordinated to $\{W_{y_k}\}_{k=1}^{n}$. It is obvious that the mapping f is continuous. Let $x \in E$. Then there exists one y_k such that $x \in D_{y_k}$ and so:

$$f(x) = y_k \in B_\delta(\phi(x)) \cap K \subseteq B_\varepsilon(\phi(x)) .$$

Suppose that $x \in G = \bigcup_{s \in S} G_s$ and let $x \in G_s$, for $s \in S$. Then from the definition of the mapping f it follows that $f(x) \in \mathrm{co}\, F_x$. Further, $x \in G_s$ implies $F_x \subset F_s$ and so:

$$f(x) \in \mathrm{co}\ F_x \subseteq \mathrm{co}\ F_s \subset B_\varepsilon(\phi(x)) .$$

Using Theorem 3 we can formulate the following fixed point theorem for multivalued mapping in paranormed space. Some fixed point theorems for multivalued mappings in paranormed spaces are proved in |3|.

THEOREM 4. *Let Y be a complete paranormed space, K be a compact and convex subset which satisfies the Zima condition* ($K \subseteq Y$), $Z \subseteq K$ *with* $\dim Z \leq 0$ *and* $\phi : K \to F(K)$ *be a lower semicontinuous mapping such that* $\phi(x)$ *is convex, for every* $x \in K \setminus Z$. *Then there exists at least one fixed point of the mapping* ϕ.

P r o o f: Since every compact topological space is normal, from Theorem 3 it follows that there exists a continuous mapping $f : K \to K$ such that all the conditions of Theorem 2 are satisfied and so there exists $x \in K$ so that $x = f(x)$. Then x is a fixed point of the mapping ϕ.

COROLLARY 1. *Let Y be a complete paranormed space, K be a closed and convex subset of* Y, $\phi : K \to F(K)$ *be a lower semicontinuous mapping such that* $\overline{\phi(K)}$ *is compact,* $\overline{\mathrm{co}}\ \phi(K)$ *satisfies the Zima condition and there exists* $M \subseteq \overline{\mathrm{co}}\ \phi(K)$ *such that* $\dim_Y M \leq 0$ *If* $\phi(x)$ *is convex, for every* $x \in \overline{\mathrm{co}}\ \phi(K) \setminus M$ *then there exists* $x \in K$ *such that* $x \in \phi(x)$.

P r o o f: We shall apply Theorem 4 taking for the set K the set $\overline{\mathrm{co}}\ \phi(K)$. Since Y is a complete paranormed space and $\overline{\phi(K)}$ is complete, it follows |6| that the set $\overline{\mathrm{co}}\ \phi(K)$ is also compact. Further, from $\phi(K) \subset K$ it follows that $\overline{\mathrm{co}}\ \phi(K) \subset K$, since K is closed and convex. This implies that $\phi(\overline{\mathrm{co}}\ \phi(K)) \subseteq \phi(K) \subseteq \overline{\mathrm{co}}\ \phi(K)$. So we

have that:

$$\phi : \overline{co}\ \phi(K) \to F(\overline{co}\ \phi(K))$$

and all the conditions of Theorem 4 are satisfied, which implies that there exists $x \in \overline{co}\ \phi(K)$ such that $x \in \phi(x)$.

COROLLARY 2. *Let Y be a complete paranormed space, K be a closed and convex subset of Y, $\phi : K \to F(K)$ be a lower semicontinuous mapping such that $\overline{co}\ \phi(K)$ satisfies the Zima condition and that the following two conditions are satisfied:*

(i) There exists a nonempty set $C \subset K$ such that $C \subset \phi(C)$ and $M \subseteq C$ with $\dim_Y M \leq 0$ so that $\phi(x)$ is convex for every $x \in K \setminus M$.

(ii) If $Q = \overline{co}\ Q \subset K$ and $Q = \overline{co}\ \phi(Q)$ then Q is compact.

Then there exists $x \in K$ so that $x \in \phi(x)$.

P r o o f: The proof is similar to the proof of the Theorem from |1|. Let $A = \{Q \mid Q \subseteq K, Q = \overline{co}\ Q, C \subseteq Q,\ \phi(Q) \subseteq Q\}$. Then $A \neq \emptyset$ and $Q \in A$ implies $\overline{co}\ \phi(Q) \in A$. Let $C_o = \bigcap_{Q \in A} Q$. Since $C \subseteq C_o$, C_o is a nonempty, closed and convex subset of K. Further, $C_o = \overline{co}\ \phi(C_o)$ and from (ii) it follows that C_o is compact. Since $M \subseteq C$ and $C \subseteq C_o$ we conclude that $M \subseteq C_o$ and $K \setminus M \supseteq C_o \setminus M$. This implies that $\phi|C_o$ satisfies all the conditions of Theorem 4 and there exists $x \in K$ such that $x \in \phi(x)$.

R e m a r k: From the proof of Theorem 3 it is easy to see that we can suppose that Y is a topological vector space, K is such that for every open neighbourhood V of zero in Y there exists an open neighbourhood U of zero in Y such that for every closed and convex subset C of K:

$$co((U + C) \cap K) \subseteq V + C$$

and ϕ is a lower semicontinuous mapping from X into K such that, for every open neighbourhood U of zero in Y the set:

$$T = \{x \mid x \in X, C' \subseteq U + \phi(x)\}$$

is open, where, C' is a compact subset of Y.

Now, we shall give an example of topological vector space Y such that for every neighbourhood U of Y there exists a neigh-

bourhood V of Y such that $co((V+C)\cap K)\subseteq C+U$, where K is a compact and convex subset of Y and C be an arbitrary closed and convex subset of K.

First, we shall give some notations and notions from |7| and |2|. A linear mapping Φ of a topological semifield E into another F is said to be positive if $\Phi(x)\geq 0$ in F, for every $x\in E$ with $x\geq 0$. Let $\|\ \|$ be a mapping of a linear space X over R into a topological semifield E and Φ be a continuous positive linear mapping of E into itself. The triplet $(X,\|\ \|,\Phi)$ is called a paranormed space over E and $\|\ \|$ a Φ-paranorm on X over E if the following conditions are satisfied:

(P1) $\|x\|\geq 0$, for every $x\in X$.

(P2) $\|\lambda x\| = |\lambda|\,\|x\|$, for every real λ and every $x\in X$.

(P3) $\|x+y\|\leq\Phi(\|x\|+\|y\|)$, for every $x,y\in X$.

DEFINITION *A set* $K, K\subseteq X$, *where* $(X,\|\ \|,\Phi)$ *is a* Φ *paranormed space, is said to be of type* Φ *if and only if for every* $n\in N$, *every* $x_1,x_2,\dots,x_n\in K-K$ *and every* $\lambda_i, 0\leq\lambda_i<1$ $(i=1,2,\dots,n)$ *such that* $\sum_{i=1}^{n}\lambda_i=1$, *we have:*

$$\left\|\sum_{i=1}^{n}\lambda_i x_i\right\|\leq\sum_{i=1}^{n}\lambda_i\Phi(\|x_i\|)$$

In |7| is proved that every topological vector space is a Φ-paranormed space over a topological semifield R_Δ. We shall use in the further text the following notation:

$$U_{\mu,\varepsilon}=\{x\mid x\in X,\ \|x\|(t)<\varepsilon,\ \text{for every } t\in\mu\}$$

where μ is a finite subset of Δ and $\varepsilon>0$. Then X is a topological vector space in which $\{U_{\mu,\varepsilon}\}_{\varepsilon>0,\mu\in\Delta}$ is the base of the fundamental system of zero in E. Similarly as in Lemma 1 we shall prove the following Lemma.

LEMMA 2. *Let* $(Y,\|\ \|,\Phi)$ *be a paranormed space, K be a compact and convex subset of Y of type* Φ. *Then for every* $U\in\{U_{\mu,\varepsilon}\}$ *there exists* $V\in\{U_{\mu,\varepsilon}\}$ *such that:*

$$\mathrm{co}((V + C) \cap K) \subseteq C + U$$

for every closed and convex subset C *of* K.

P r o o f: Let $U = U_{\mu,\varepsilon}$, $\mu \subset \Delta$ and $\varepsilon > 0$. Since the mapping Φ is linear and continuous it follows that $N_1 = (\Phi^2)^{-1}(U_{\mu,\varepsilon})$ is a neighbourhood of zero in R_Δ and let $\mu' \subset \Delta$ and $\varepsilon' > 0$ be such that:

$$U_{\mu',\varepsilon'} \subseteq \{x \mid \|x\| \in N_1\}.$$

Let $\mu'' \subset \Delta$ and $\varepsilon'' > 0$ be such that:

$$U_{\mu'',\varepsilon''} + U_{\mu'',\varepsilon''} \subseteq U_{\mu',\varepsilon'} .$$

We shall prove that $\mathrm{co}((C + U_{\mu'',\varepsilon''}) \cap K)) \subseteq C + U_{\mu,\varepsilon}$. Let z and c be as in Lemma 2, where we take $U_{\mu'',\varepsilon''}$ instead of U_δ and $U_{\mu',\varepsilon'}$ instead of $U_{\frac{\varepsilon}{C^2(K)}}$. Then we have that $t \in \mu$ implies:

$$\|z-c\|(t) = \Big\|\sum_{j=1}^{m} \gamma_j z_j - \sum_{j=1}^{m} \gamma_j c_j\Big\|(t) \leq$$

$$\leq \sum_{j=1}^{m} \gamma_j \Phi(\|z_j - c_j\|)(t) =$$

$$= \sum_{j=1}^{m} \gamma_j\, \Phi\Big(\sum_{k=1}^{n} \beta_k(z_j)\, \Phi(\|z_j - x_k\|)\Big)(t) \leq$$

$$\leq \sum_{j=1}^{m} \gamma_j \Big(\sum_{k=1}^{n} \beta_k(z_j)\, \Phi^2(\|z_j - x_k\|)(t)\Big) =$$

$$= \sum_{j=1}^{m} \gamma_j \Big(\sum_{\substack{k=1 \\ \beta_k(z_j) \neq 0}}^{n} \beta_k(z_j)\, \Phi^2(\|z_j - x_k\|)(t)\Big)$$

Since $\beta_k(z_j) \neq 0$ implies that $z_j - x_k \in U_{\mu',\varepsilon'}$ and so $\|z_j - x_k\| \in N_1$ we have that

$$\|z-c\|(t) \leq \sum_{j=1}^{m} \gamma_j \sum_{\substack{k=1 \\ \beta_k(z_j) \neq 0}}^{n} \beta_k(z_j)\varepsilon = \varepsilon \text{ and so } z-c \in U_{\mu,\varepsilon} .$$

Using this Lemma we can formulate the following fixed point theorem.

THEOREM 5. *Let* Y *be a complete* Φ *paranormed space,* K *be a compact, convex subset of type* Φ *of* Y *and* $Z \subset K$ *with* $\dim_Y Z \leq 0$. *Further, let* $\phi: K \to F(K)$ *be a lower semicontinuous mapping such that* $\phi(x)$ *is convex, for every* $x \in K \setminus Z$ *and the set* $\{x | x \in K, C \subseteq U + \phi(x)\}$ *is open, for every* $C \subseteq Y$ *be compact and* U *be an arbitrary open neighbourhood of zero in* Y. *Then there exists at least one fixed point of the mapping* ϕ.

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REZIME

TEOREMA O OSOBINI GOTOVO NEPREKIDNE SELEKCIJE
I PRIMENA

U ovom radu dokazano je uopštenje teoreme Michaela i Pixleyao gotovo neprekidnoj selekciji u paranormiranim prostorima.

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A NOTE ON THE SPACE $\mathcal{D}'_A$

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1. In |4| a subspace $\mathcal{D}'_A \subset \mathcal{D}'$ is investigated. It is shown that the elements of $\mathcal{D}'_A$ may be uniquely expanded into a Hermitte series. First the space A is defined as the space of functions defined on R such that $f \in A$ iff for some $a > 4$, $F(x)e^{-x^2/a} \in L_2(R)$ Furthermore , the set $\mathcal{D}'_A$ is defined such that $f \in \mathcal{D}'_A$ iff there exist $k \in N_o$, $F \in A$ such that

(1) $$f = D^k(F)$$

where D is the tempered derivative given in |1| with

(2) $$D^k(F) = \exp(-x^2/4)(\exp/x^2/4)F)^{(k)} \quad , \quad D^o(F) = F$$

and k-th derivative is in the distributional sense. It is shown that $\mathcal{D}'_A$ is a subspace of $\mathcal{D}'$, and an estimate of Hermitte coefficients of elements from $\mathcal{D}'_A$ is given.

In this paper, using the elementary approach, we shall characterize the elements from $\mathcal{D}'_A$. Also, we shall introduce the convergence in $\mathcal{D}'_A$ and characterize it. In this way we shall show that $\mathcal{D}'_A$ is identical with the space of the $K'(M_p)$- type, introduced in |2|, for the special sequence (M_p).

2. Let us put

(3) $$M_p(x) = e^{x^2/(4-1/p)} \quad , \quad p=1,2,\ldots .$$

THEOREM 1. $f \in \mathcal{D}'_A$ *iff there exist* $p \in N, m \in N_o$ *and bounded measurable functions* f_j, $0 \le j \le m$, *such that*

$$(4) \qquad f = \sum_{j=o}^{m} (M_p f_j)^{(j)} .$$

P r o o f. Let $f \in \mathcal{D}'_A$. This means that there exist $k \in N_o$ and $F \in A$ such that (1) holds.

The tempered integral given in |1| is defined on the space of locally integrable functions by

$$(5) \quad S^o(G) = G; \quad S(G) = e^{-x^2/4} \int_o^x e^{t^2/4} G(t)dt; \quad S^k(G) = S(S^{k-1}G)$$

As $A \subset L_1^{loc}$, if we put $F_1(x) = S(F(x))$, then

$$|F_1(x)| \le e^{-x^2/4} | \int_o^x F(t)\, e^{-t^2/a}\, e^{+t^2/a}\, e^{+t^2/4}\, dt| \le$$

$$\le e^{-x^2/4} (\int_o^x (|F(t)|e^{-t^2/a})^2 dt)^{1/2} (\int_o^x e^{2(t^2/a+t^2/4)} dt)^{1/2} \le$$

$$\le K\sqrt{x}\; e^{x^2/a} \quad \text{where} \quad K = (\int_R (|F(t)|e^{-t^2/a})^2 dt)^{1/2} .$$

There exists $p \in N$ such that $a > 4 + 1/p$. This implies that for a suitable constant K_o

$$|F_1(x)| \le K_o M_p(x) \text{ i.e. } M_{p+1}^{-1}(x) F_1(x) \in L_2$$

Since $D(f) = f' + x \cdot f/2$ and $D(S(F)) = F$, the members of the expression $D^{k+1}(F_1) = f$ are of the form $c_{j,\ell} x^j (F_1(x))^{(\ell)}$, $j \le k+1$, $\ell \le k+1$ and $c_{j,\ell}$ are real numbers.

After using Leibnitz's formula we obtain that in $D^{k+1}(F_1)$ the members are of the form $\tilde{c}_{j,\ell}(x^j F_1(x))^{(\ell)}$ with some other coefficients $\tilde{c}_{j,\ell}$. In the same way as for $F_1(x)$, we can prove that $|x^j F_1(x)| \le K_j \cdot M_p(x)$. By putting

$$f_\ell = \sum_{j \le k+1} \tilde{c}_{j,\ell} x^j F_1(x) \quad \text{we get}$$

$$f(x) = \sum_{\ell \leq k+1} (M_{p+1}(x) f_\ell(x))^{(\ell)}$$

which means that f has the representation of the form (4).

Let us now assume that f has the representation of the form (4). We denote $M_p f_j = F_j$. Since $M^{-1}_{p+1} F_j \in L_2$, similarly as in the first part of that proof, it follows that $S(x^r F_j(x) M^{-1}_{p+3}(x)) \in \in L_2$ for any $r \in N_o$ From the formula

(6) $\qquad f' = D(f) - x \cdot f/2$ and $x' D(f) = D(xf) - f$,

taking operator S enough times, we obtain that

$$\sum_{j \leq m} (F_j(x))^{(j)} = D^k(H(x))$$

for some $k \in N$, and some function $H(x)$ such that

$$M^{-1}_{p+p_o}(x) H(x) \in L_2 .$$

The integer p_o depends on the number of applications of operator S. So we have proved that $f \in \mathcal{D}'_A$.

Let us define the convergence structure in the space $\mathcal{D}'_A$.

We say that the sequence (f_n) from $\mathcal{D}'_A$ converges to $f \in \mathcal{D}'_A$ iff there exist a sequence of functions (F_n), a function F, $k \in N_o$ and $a > 0$ such that

(7) $\qquad D^k(F_n) = f_n$, $D^k(F) = f$

and $(e^{-x^2/a} F_n(x))$ is the sequence from L_2 which converges to $e^{-x^2/a} F(x)$ in the L_2 norm.

THEOREM 2. *The sequence* (f_n) *from* $\mathcal{D}'_A$ *converges to* $f \in \mathcal{D}'_A$ *iff there exist* $p \in N, m \in N_o$, *sequences of bounded measurable functions* $(F_{j,n})$, , $0 \leq j \leq m$, *and bounded measurable functions* F_j, $0 \leq j \leq m$, *such that*

$$F_{j,n} \to F_j \qquad \textit{almost everywhere}$$

(8) $$f_n = \sum_{j\le m} (M_p F_{j,n})^{(j)} \qquad f = \sum_{j\le m} (M_p F_j)^{(j)} .$$

P r o o f. If condition (7) is satisfied we put $F_{1,n} = S(F_n)$ and $F_1 = S(F)$. From the estimate

$$|F_{1,n}(x)-F_1(x)| \le e^{-x^2/4}\left(\int_0^x |F_{1,n}(t)-F_1(t)|^2 e^{-2t^2/a}dt\right)^{1/2} \cdot \left(\int_0^x e^{2(t^2/a+t^2/4)}dt\right)^{1/2}$$

it follows that $(F_{1,n}(x))$ converges almost everywhere to F_1. Similarly, as in Theorem 1, observing the expressions $D^{k+1}F_{1,n}$ and $D^{k+1}F_1$ we can prove that (8) holds.

Let us suppose that (8) holds. If we denote

$$F_j = M_p(x)\cdot f_j \quad \text{and} \quad F_{j,n} = M_p(x)\cdot f_{j,n}$$

similarly as in the proof of Theorem 1, we can show that

$$x^r F_{j,n} M_{p+3}^{-1}(x) \in L_2 \quad \text{and} \quad x^r F_j M_{p+3}^{-1}(x) \in L_2 ,$$

$0 \le j \le m$; $r \in N_o$. From the Lebesque theorem it follows that

$$x^r F_{j,n} M_{p+3}^{-1}(x) \xrightarrow{L_2} x^r F_j M_{p+3}^{-1}(x),$$

In the same way as in the proof of Theorem 1, using the formulae of (6) and applying S sufficiently many times we get the assertion.

As the tempered derivative D characterizes the space S' which is proved in |1|, from Theorems 1. and 2. we get

COROLLARY. *(i)* $f \in \mathcal{D}'_A$ *iff for some* $a > 4$, $m \in N_o$ *and a bounded continuous function* $F(x)$

$$f = (e^{x^2/a} F(x))^{(m)}$$

(ii) The sequence $(f_n) \in \mathcal{D}'_A$ *converges to* $f \in \mathcal{D}'_A$ *iff there exist bounded continuous functions* F_n, $n \in N$, *a bounded continuous function* F, $a > 4$, *and* $m \in N_o$, *such that*

$$f_n = (e^{x^2/a}F_n(x))^{(m)}\ ,\quad f(x) = (e^{x^2/a}F(x))^{(m)}$$

and $(F_n(x))$ *converges to* $F(x)$ *almost uniformly.*

P r o o f. We only have to apply the operator S sufficiently many times in (4) and (8).

3. In |2| the spaces of the type $K'(M_p)$ are introduced and investigated.

Using the theory from |2| and |3|, from (4) and (8) directly it follows that spaces $\mathcal{D}_A'$ and $K'(\exp(-x^2/(4+1/p))$ are indentical both in a set theoretical and a topological sense.

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REZIME

PRILOZI TEORIJI PROSTORA $\mathcal{D}_A'$

U ovom radu, koristeći elementarni pristup, karakterišemo elemente iz $\mathcal{D}_A'$ uvedenog u |4|. Takodje uvodimo konvergenciju u prostor $\mathcal{D}_A'$ i karakterišemo je. Tako pokazujemo da je $\mathcal{D}_A'$ identičan sa prostorom $K'(M)$- tipa uvedenog u |2|, za specijalan niz $(M_p(x))$.

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SOLVABILITY OF CONVOLUTION EQUATIONS IN $H'\{M_p\}$

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ABSTRACT.

We obtain estimates for the Fourier transform of convolutors on the space $H'\{M_p\}$ introduced in |7|. This enables us to prove that the convolution equation (1) in $H'\{M_p\}$ is solvable in $H'\{M_p\}$ iff it is solvable in $K'(M_p)$ for each $p \geq p_o(V)$.

INTRODUCTION

We introduced the space $H'\{M_p\}$ in |7|. We showed that $H'\{M_p\}$ can be obtained as the inductive limit of the spaces $H'(M_p)$, p=1,2,..., defined in |10|. Some examples of the space $H'\{M_p\}$ were analysed in: |10| and |5| for $M_p(x) := M(px)$ where $M(x)$ is a fixed convex function; in |1|, |8|, |9| and |4| for $M_p(x) := p\cdot|x|^s$ where s is a fixed natural number; and in |6| for $M_p(x) := |x|^p$.

In the third part of this paper we prove that the convolution equation

(1) $\qquad S * U = V$,

where S belongs to the space of convolutors on $H'\{M_p\}$, denoted by $O_c'(H'\{M_p\})$, is solvable by U in $H'\{M_p\}$ for arbitrary $V \in H'\{M_p\}$ iff it is solvable in each $K'(M_p)$, $p \geq p_o(V) \in N$. We obtain this result from some assetrions in |7| (here given in the first part of the paper), and from the estimates for the Fourier transform of the convolutor S (proved in the second part of the paper). Of course, we use the well - known results on surjectivity of equation (1) given in |1|, |9| and |5|.

As in |7| we are considering the one dimensional case, but with some simple modifications the results of this paper can be used for the n-dimensional case.

1. SOME NOTIONS AND ASSERTIONS FROM |7|

Throughout the paper we denote by $\{m_p(x)\}_{p\in N}$, $x \geq 0$, a sequence of continuous increasing functions for $x \geq 0$ which satisfy $m_p(0)=0$, $m_p(\infty)=\infty$ and $m_p(x) \leq m_{p+1}(x)$ for each $p=1,2,\ldots$ and $x \geq 0$. Putting

$$\text{(2)} \qquad M_p(x) := \int_0^{|x|} m_p(t)\,dt \ , \ p=1,2,\ldots, \ x \in R$$

we obtain another sequence of functions. Each $M_p(x)$ is an even convex function and increases to infinity faster than any linear function when $|x| \to \infty$. This implies that its dual function in the sense of Young (|3|)

$$M_p^*(y) = \int_0^{|y|} m_p^{-1}(t)\,dt$$

is finite for arbitrary $y \in R$; $m_p^{-1}(x)$, $x \geq 0$, is the inverse function for $m_p(x)$.

Our main assumption on the sequence $\{M_p(x)\}_{p\in N}$ is

(A) For each $p \in N$ there exist $X_p \geq 0$ and $p' \in N$ such that $M_p(px) \leq M_{p'}(x)$ for $|x| \geq X_p$.

Let us denote the smallest p' for which this inequality holds for large $|x|$ by $r(p)$. Observe that this condition is satisfied in the mentioned spaces of the type $H'\{M_p\}$.

DEFINITION 1. *The vector space of smooth functions* $\phi(x)$ *on* R *with the properity*

$$\gamma_p(\phi) := \sup\{|\phi^{(j)}(x)|\cdot\exp(M_p(x));\ x \in R,\ 0 \leq j \leq p\} < \infty$$

for each $p \in N$ *, topologized with the sequence of norms* $\{\gamma_p\}_{p\in N}$ *is denoted by* $H\{M_p\}$.

$H\{M_p\}$ is a space of the type $K\{\exp(M_p(x))\}$ from |2|. The dual of $H\{M_p\}$, denoted by $H'\{M_p\}$, is a proper subspace of the space of distributions $\mathcal{D}'$.

Following |10| and |5|, we denote by $K(M_p)$ the space of smooth functions $\phi(x)$ on R with the property

$$\rho_{p,k}(\phi) := \sup\{|\phi^{(j)}(x)|\cdot\exp(M_p(kx)); x \in R,\ 0 \le j \le k\} < \infty$$

for each k=1,2,... and fixed $p \in N$. We have

THEOREM 1. *The spaces* $H\{M_p\}$ *and* $proj K(M_p)$ *are topologically isomorphic. The spaces* $H'\{M_p\}$ *and* $ind K'(M_p)$ *are topologically isomorphic when* $H'\{M_p\}$ *and each* $K'(M_p)$ *are endowed with strong topology.*

Naturally, $proj K(M_p)$ stands for the projective limit of the spaces $K(M_p)$; an analogous meaning has $ind K'(M_p)$.

The convolution between $S \in H'\{M_p\}$ and $\phi \in H\{M_p\}$ is defined in the usual way

$$(S * \phi)(x) := \langle S(y), \phi(x-y) \rangle$$

and it is a smooth function which defines a regular element from $H'\{M_p\}$. We are mainly interested in those distributions S from $H'\{M_p\}$ for which the function $(S * \phi)(x)$ is in $H\{M_p\}$ whenever $\phi(x)$ is from $H\{M_p\}$.

DEFINITION 2. *The distribution* $S \in H'\{M_p\}$ *is a convolution operator - convolutor iff the mapping* $S * : \phi \to S * \phi$ *is continuous and maps* $H\{M_p\}$ *into itself.*

We denote the space of convolutors on $H'\{M_p\}$ by $O'(H'\{M_p\})$. It is known that if $1 \le p \le q$ then $K'(M_p) \subset K'(M_q)$ and $O_c'(K'(M_q)) \subset O_c'(K'(M_p))$.

THEOREM 2. *The distribution* $S \in H'\{M_p\}$ *is a convolutor on* $H'\{M_p\}$ *iff for each* $p \in N$ *there exist* $m \in N_0$ *and a continuous function on* R, $F(x)$, *with the property*

$$\|F(x) \cdot \exp(m_p(x))\|_{L\infty} < \infty \text{ , such that } S(x) = D^m F(x) \text{ .}$$

The symbol "D" stands for the distributional derivative. Theorem 2 together with the representation of the convolutors on $K'(M_p)$ for fixed $p \in N$ (see |10|) implies

$$(3) \qquad \bigcap_{p=1}^{\infty} O_c'(K'(M_p)) = O_c'(H'\{M_p\}).$$

This set - theoretical equality will be essential in the proof of Theorem 6.

2. THE FOURIER TRANSFORMATION ON $H'\{M_p\}$

The Fourier transformation $\hat{\phi}$ of $\phi \in H\{M_p\}$ defined by

$$(\mathcal{F}\phi(x))(\zeta) := \hat{\phi}(\zeta) := \int_R \exp(-ix\zeta)\cdot\phi(x)dx$$

is an entire analytic function of the complex variable ζ . Let us denote by $\mathrm{H}\{M_p\}$ the set of entire analytic functions $\psi(\zeta)$ with the property $\mathcal{F}\phi = \psi$ for some $\phi \in H\{M_p\}$. In Theorem 3 we shall prove that the Fourier transformation is a topological isomorphism from $H\{M_p\}$ onto $\mathrm{H}\{M_p\}$. In order to characterize $\mathrm{H}\{M_p\}$, we shall use the following normed spaces introduced in |10| :

$$W^k_{M,A} := \{\phi \in C^\infty \,|\, \sup\{|\phi^{(j)}(x)|\cdot\exp(M(x/A));\ \ x \in R,$$

$$0 \le j \le k\} < \infty\},$$

$$W_k^{M,A} := \{\psi \in \mathcal{U} \,|\, \sup\{(1+|x|)^k\cdot|\psi(x+iy)|\cdot\exp(-M(Ay))\ ;$$

$$x + iy \in C\} < \infty\}$$

where M(x) is a convex function of the from (2), k is a non-negative integer, A a positive constant, C^∞ is the space of smooth functions on R and U is the space of entire analytic functions on C.

We shall also need the normed space $H(M_p)$, the space of smooth functions $\phi(x)$ on R such that $\gamma_p(\phi) < \infty$ for fixed $p \in N$. Observe that $H(M_p) = W^p_{M_p,1}$.

From the proof of Theorems 1 and 2 in |3| , page 20, the following inclusions hold:

$$(4) \qquad F(W^k_{M,A}) \subset W^{M,A+d}_k ,$$

$$(5) \qquad F(W^{M,A}_{k+2}) \subset W^k_{M,A+d}$$

for arbitrary $d > 0$. Let us prove

LEMMA 1. *The following equalities hold both in the set - theoretical and topological sence:*

$$(a) \qquad \bigcap_{p=1}^{\infty} W_p^{M^*_p,1+d} = \bigcap_{p=1}^{\infty} W_p^{M^*_p,1}$$

$$(b) \qquad \bigcap_{p=1}^{\infty} W^p_{M_{p+2},1+d} = \bigcap_{p=1}^{\infty} W^p_{M_p,1}$$

for arbitrary $d > 0$.

P r o o f. We shall prove only part a), since the proof of part b) is similar to that of a).

It is clear that $W_p^{M^*_p,1} \subset W_p^{M^*_p,1+d}$ hence $\bigcap_{p=1}^{\infty} W_p^{M^*_p,1} \subset$ $\subset \bigcap_{p=1}^{\infty} W_p^{M^*_p,1+d}$ also in the topological sence.

In order to prove the opposite inclusion, let us show that there exists $p_1 = p_1(p,d) \in N$ for given $p \in N$ and $d > 0$, such

that for sufficiently large $|y|$

(6) $$-M^*(y) \leq -M^*_{p_1}((1+d)y) .$$

This inequality implies a) in a set - theoretical sence; if the mentioned spaces are endowed with the projective topology, using the same inequality one obtaines a) also in a topological sence. So, let p and d be given. From condition (A) follows the existence of $p_1 \in N$ such that for sufficiently large $|x|: M_p((1+d)x) \leq M_{p_1}(x)$. Turning to the dual functions in the sense of Young, we obtain (6).

Since in the set - theoretical sense

$$H\{M_p\} = \bigcap_{p=1}^{\infty} H(M_p)$$

we at once get

(7) $$H\{M_p\} = F\left(\bigcap_{p=1}^{\infty} H(M_p)\right) \subset \bigcap_{p=1}^{\infty} F(H(M_p))$$

Let us prove that the inclusion in (7) can be replaced by the equality.

LEMMA 2. *We have in the set - theoretical sence*

$$H\{M_p\} = \bigcap_{p=1}^{\infty} F(H(M_p)) .$$

P r o o f. Let $\psi \in \bigcap_{p=1}^{\infty} F(H(M_p))$; from (4) we obtain that $\psi(\zeta)$ is an entire analytic finction and that it insreases on the $\xi(:=\operatorname{Re}\zeta)$ - axis faster than any power of $1/|\xi|$. Its inverse Fourier transformation.

(8) $$(F^{-1}\psi(\xi))(x) := \phi(x) := \frac{1}{2\cdot\pi}\cdot\int_R \exp(ix\xi)\cdot\psi(\xi)d\xi$$

is a smooth function on R. From (5) and (8) we get

$$2\pi\cdot\phi(x) = (F\psi(-\xi))(x) \in \bigcap_{p=1}^{\infty} W^p_{M_{p+2},1+2d} .$$

Hence by Lemma 1, part b), we obtain $2\pi\cdot\phi(x) \in H\{M_p\}$, and this implies $\psi(\zeta) \in H\{M_p\}$.

We can now prove

THEOREM 3. *a) The elements from* $H\{M_p\}$ *are entire analytic functions* $\psi(\zeta)$ *which satisfy*

$$h_p(\psi) := \sup\{(1+|\xi|)^p\cdot|\phi(\xi+i\eta)|\cdot\exp(-M_p^*(\eta));\ \xi+i\eta \in C\} < \infty$$

for each $p \in N$.

b) If the topology on $H\{M_p\}$ *is given by the set of seminorms* $\{h_p\}_{p\in N}$, *the Fourier transformation is a topological isomorphism from* $H\{M_p\}$ *onto* $H\{M_p\}$.

P r o o f. a) Follows from Lemmas 1 and 2.
b) The space $H\{M_p\}$ is of the type

$Z\{(1+|\xi|)^p\cdot\exp(-M^*_p(\eta))\}$ introduced in |2|, hence it is a Frechet space. Since the Fourier transformation is a surjective mapping from $H\{M_p\}$ onto $H\{M_p\}$ by its definition, we can use the open mapping theorem, which asserts just what we want to prove.

The dual space $H'\{M_p\}$ of $H\{M_p\}$ is the space of Fourier transformations of the distributions from $H'\{M_p\}$ defined by the Parseval formula

$$\langle FS, F\phi\rangle := 2\cdot\pi\langle S, \check{\phi}\rangle$$

where $S \in H'\{M_p\}$, $\phi \in H\{M_p\}$ and $\check{\phi}(x) := \phi(-x)$.

In the case when S is a convolutor, we have the following

THEOREM 4. *The Fourier transformation of* $S \in O_c'(H\{M_p\})$ *denoted by* $\hat{S}(\xi)$ *is a function which can be analitically continued on the whole complex plane* C *and it has the following property: for each* $p \in N$ *there exists a positive*

number c *and a natural number* n *so that*

$$(9) \qquad |\hat{S}(\xi+i\eta)| \leq c\cdot(1+|\xi|)^n\cdot\exp(M_p^*(\eta)).$$

Conversely, if for an entire analytic function $\hat{S}(\zeta)$ *for each* $p\in N$ *there exist* $c>0$ *and* $n\in N$ *so that (9) holds, then there exists a convolutor* S *on* $H'\{M_p\}$ *such that* $FS=\hat{S}$.

P r o o f. Let $S\in H'\{M_p\}$. From Theorem 2 follows that $\hat{S}(\zeta) = (i\zeta)^m\,\hat{F}(\zeta)$ where for given $p\in N$, m and F(x) are chosen as in Theorem 2. Observe that the rate on increase in infinity of the function F(x) implies that $\hat{F}(\zeta) := (FF)(\zeta)$ is an entire analytic function. From |3|, page 21 follows

$$|\hat{F}(\xi+i\eta)| \leq c_1\cdot\exp(M_p^*((1+d)\eta))$$

for some $c_1=c_1(d,p)$, $0<d<1$. Using condition (A) we can choose $p_1\in N$, $p_1<p$ (except maybe for finitely many) so that for sufficiently large $|\eta|$

$$M_p^*((1+d)\eta) \leq M_{p_1}^*(\eta)$$

and this implies the estimate (9) for p_1 in the place of p. We can choose p so that the corresponding p_1 come across a sub-sequence of the sequence of natural numbers and this observation finishes the proof of necessity of condition (9).

Let us suppose now that $\hat{S}(\zeta)$ is an entire analytic function which satisfies (9). From |10|, Bemerkung IV.2, it follows that $S(x) = \frac{1}{2\cdot\pi}\,(\hat{S}(-\xi))(x)$ is the finite sum of the distribution derivatives of continuous functions $F_j(x)$, i.e.

$$(10) \qquad S(x) = \sum_{j=1}^{m} D^j F_j(x) \qquad \text{where}$$

$F_j(x) = O(\exp(-M_p(kx)))$ when $|x|\to\infty$ and $k>0$ does not depend on j.

Again using condition (A), we can find a suitable $p_1\in N$, $p_1<p$, such that

$$F_j(x) = O(\exp(-M_{p_1}(x))) \quad \text{when} \quad |x| \to \infty \text{ for each } j,\ 0 \le j \le m.$$

Integrating, if necessary, each term in (10) sufficiently many times, we can reduce the sum in (10) to one single term, i.e.

$$S(x) = D^m F(x)$$

where $F(x)$ is continuous function on R such that

$$F(x) = O(\exp(-M_{p_1}(x))) \quad \text{when} \quad |x| \to \infty .$$

As in the first part of the proof, we can choose p such that the corresponding p_1 come across a sub-sequence of the sequence of natural numbers.

So, we have proven Theorem 4 for each $p \in N$ except maybe for the first finitely many; but if it holds for some p, then it holds for each p' smaller than p.

Theorem 4 implies that if S is a convolutor on $H'\{M_p\}$ then the mapping $\hat{S} * : \psi \to \hat{S} * \psi$ $(\hat{S} := \mathcal{F} S)$ is a continuous linear mapping from $H\{M_p\}$ into $H\{M_p\}$. Hence, if $T \in H'\{M_p\}$, one can define the product $\hat{S} \cdot \hat{T}$ by

$$< \hat{S} \cdot \hat{T}, \psi > \ := \ < \hat{T}, \hat{S} \cdot \psi > \quad \text{where} \quad \hat{T} := \mathcal{F} T \quad \text{and } \psi \in H\{M_p\} .$$

It is easy to prove that $\mathcal{F}(S * T) = \mathcal{F}S \cdot \mathcal{F}T$.

3. SOLVABILITY OF (1) IN $H'\{M_p\}$

Our task is to characterize the surjective convolutors on $H'\{M_p\}$, and, what turns out to be equivalent with surjectivity, to find those convolutors on $H'\{M_p\}$ which have fundamental solutions in $H'\{M_p\}$. Various "slowly decreasing functions" play an important role in this part. In |5| the following definition is given:

DEFINITION 3. *An entire analytic function* $F(\zeta)$ *is called an* M_q *-slowly decreasing function* $(q \in N)$ *if it satis-*

fies an inequality of the form

$$(11) \qquad \sup\{|F(\xi+w)|;\ |w| \le \rho(\log(1+|\xi|))\ ;\ w \in R\} \ge \ge C_o \cdot (1+|\xi|)^{-N_o}, \quad \xi \in R,$$

for some positive constants C_o *and* N_o, *and*

$$(12) \qquad \rho(x) := A \cdot \frac{x}{M_q^{-1}(x)} + B, \quad x > 0$$

for some $A > 0$ *and* $B \in R$.

If (11) holds for $\rho(x) = \text{const}$, $F(\zeta)$ *is called extremely slowly decreasing.*

It is easy to show that $\frac{x}{M_q^{-1}(x)} \le M_q^{*-1}(x)$ for $x > 0$.

Since this function tends to infinity when x does, there exist positive numbers A_1 and L_1 such that

$$(13) \qquad \rho(x) \le \rho_1(x) := A_1 \cdot M_q^{*-1}(x) \quad \text{for } x \ge L_1.$$

The sign "-1" stands for the inverse function.

The following theorem gives a sufficient condition for an entire analytic function to be extremely slowly decreasing.

THEOREM 5. *Let* $F(\zeta)$ *be an entire analytic function which is* M_q *-slowly decreasing for some* $q \in N$. *Let* $p \in N$ *be larger than* $r(\max\{|A_1|, q\})$, (A_1 *from (13)) and let us suppose that* $F(\zeta)$ *satisfies an estimate (9) for some* c, n *and this* p. *Then* $F(\zeta)$ *is extremely slowly desreasing.*

P r o o f. The property of p implies that the number

$$A_2 := \sup\{\frac{M_p^*(x)}{M_q^*(x/A_1)},\ x \ge L_1\} + 1$$

is finite. Let us take $L \ge L_1$ so large that $\rho_1(\log(1+|\xi|)) > 1$ for each ξ with $|\xi| > L$. Let us fix ξ with $|\xi| > L$ and define

$$\beta := \frac{\log \rho}{\log(M_p^{*-1}(A_2 \cdot M_q^*(\rho/A_1))) - \log \rho}$$

where $\rho := \rho_1(\log(1+|\xi|)) > 1$. The definition of A_2 implies $\beta > 0$ and let us put

$$\bar{R} := \rho^{\frac{\beta+1}{\beta}}$$

As in |4| , we apply Hadamard's Three Circles Theorem on the function $F(\xi+\lambda w)$ (λ- complex variable) for the circles with radiuses $1, \rho, \bar{R}$ and $\gamma := \frac{\log(\bar{R}/\rho)}{\log \bar{R}} = \frac{1}{\beta+1}$. All the time, w is a complex parameter. So we have

(14) $$\sup\{|F(\xi+w)| \; ; \; |w| \leq 1\} \geq$$
$$\geq (\sup\{|F(\xi+\rho w)| \; ; \; |w| \geq 1\})^{1+\beta} / (\sup\{|F(\xi+\bar{R}w)|; \; |w| \leq 1\})^{\beta}.$$

Using (9) we obtain

$$|F(\xi+\bar{R}w)| = |F(\xi+\bar{R}\cdot \mathrm{Re}w + i\cdot\bar{R}\cdot \mathrm{Im}w)| \leq$$
$$\leq c\cdot(1+|\xi|)^n\cdot(1+\bar{R})^n\cdot\exp(M_p^*(\bar{R})) \leq c'\cdot c\cdot(1+|\xi|)^n\cdot\exp(2\cdot M_p^*(\bar{R}))$$

where we have put $c' := \sup\{(1+\bar{R})^n\cdot\exp(-M_p^*(\bar{R})) \; ; \; \bar{R} \in R\} < \infty$

Since we have constructed $\bar{R}$ so that $M_p^*(\bar{R}) = A_2\cdot M_q^*(\rho/A_1)$ we have

(15) $$\sup\{|F(\xi+\bar{R}w)| \; ; \; |w| \leq 1\} \leq C\cdot(1+|\xi|)^{n+A_2}$$

for some $C > 0$. Returning to (14) using (11) we obtain the statement for $|\xi| \geq L$.

Using the Maximum Principle we obtain for $|\xi| \leq L$

$$\sup\{|F(\xi+w)| \; ; \; |w| \leq 1\} \geq C_1 > 0$$

and this together with (15) gives

$$\sup\{|F(\xi+w)| \; ; \; |w| \leq 1\} \geq C_2 \; (1+|\xi|)^{-(N_o+n+2A_2)}$$

i.e. $F(\zeta)$ is extremely slowly decreasing.

If we suppose that instead of the condition (A) the stronger condition

(A′) Let $p, p' \in N$ and $p' > p$. For each $\bar{C} > 0$ there exists $X_p > 0$ such that $M_p(\bar{C}\cdot x) \leq M_{p'}(x)$ for $|x| \geq X_p$

is satisfied, then in the same way as Theorem 5 we may prove the following Theorem 5′, which generalized Theorem 3 from |4|.

THEOREM 5′. *Let* $F(\zeta)$ *be an entire analytic function which satisfies an estimate (9) for some* c, n *and* p. *If* $F(\zeta)$ *is* M_q *- slowly decreasing for some natural number* $q, 1 \leq q < p$, *then it is extremely slowly decreasing.*

Theorems 4 and 5 combined with relation (3) imply

THEOREM 6. *If the Fourier transform* $\hat{S}$ *of the distribution* $S \in O_c'(H'\{M_p\})$ *is* M_q *-slowly decreasing for some* $q \in N$ *then* $\hat{S}$ *is* M_p *-slowly decreasing for each* $p \in N$.

Let us prove now

THEOREM 7. *The following conditions are equivalent, provided that* $S \in H'\{M_p\}$

(s_1) $\hat{S}$ *is* M_q*-slowly decreasing for some* $p \in N$, ($\hat{S} := FS$) ;

(s_2) S *has a fundamental solution in* $H'\{M_p\}$;

(s_3) $S * H'\{M_p\} = H'\{M_p\}$.

P r o o f. Since Dirac′s measure is in $H'\{M_p\}$, we have (s_3) => (s_2). If S has a fundamental solution in $H'\{M_p\}$, in view of Theorem 1 it belongs to some $K'(M_p)$. The Theorem in |5|, page 2, states, among other things, that the convolutor S on $K'(M_p)$ has a fundamental solution in $K'(M_p)$ iff $\hat{S}$ is a M_p -slowly decreasing function. Hence (s_2) => (s_1). Finally, if $\hat{S}$ is M_p -slowly decreasing for some $p \in N$, by Theorem 6 and the mentioned theorem from |5|, it is surjective on $K'(M_p)$ for each p=1,2,... .But,by Theorem 1 the union of the spaces $K'(M_p)$

is just $H'\{M_p\}$. i.e. $(s_1) \Rightarrow (s_3)$.

Let us turn to the convolution equation (1). We suppose that it is a convolutor on $O_c'(H'\{M_p\})$, hence by (3) it is a convolutor on each space $K'(M_p)$. If X' is one of the spaces $H'\{M_p\}$ or $K'(M_p)$, $p=1,2,\ldots$, we say that (1) is solvable in X' iff for each $V \in X'$ there exists an $U \in X'$ so that (1) holds.

THEOREM 8. *The convolution equation (1) is solvable in $H'\{M_p\}$ iff it is solvable in each $K'(M_p)$, $p=1,2,\ldots$*

P r o o f. Let $V \in H'\{M_p\}$ be given and let us denote by p_o the smallest integer for which $V \in K'(M_p)$ (see Theorem 1). If (1) is solvable in $H'\{M_p\}$, the implication $(s_3) \Rightarrow$ $\Rightarrow (s_1)$ from Theorem 7 shows that $S := FS$ is M_p -slowly decreasing for some $p \in N$, and by Theorem 6 it is M_p -slowly decreasing for each $p \in N$. This implies that (1) is solvable in $K'(M_p)$ for each $p \geq p_o$. The converse is obvious in view of the implication $(s_1) \Rightarrow (s_3)$.

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REZIME

REŠIVOST KONVOLUCIONIH JEDNAČINA U $H'\{M_p\}$

U radu su dati potrebni i dovoljni uslovi za rešivost konvolucione jednačine

$$S * U = V$$

u prostoru $H'\{M_p\}$ (|7|).

Dokazana je teorema.

TEOREMA. Neka je S konvolutor na $H'\{M_p\}$. Sledeći uslovi su ekvivalentni:

(a) Preslikavanje $S*: H'\{M_p\} \to H'\{M_p\}$ je surjektivno ;

(b) S ima fundamentalno rešenje u $H'\{M_p\}$;

(c) Furijeova transformacija konvolutora S (koja je cela analitička funkcija) je M_p- sporo opadajuća funkcija za neko $p \in N$.

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ON SEMIGROUP VALUED ADDITIVE EXHAUSTIVE SET FUNCTIONS

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1. INTRODUCTION

Using the functional from |4| defined on a semigroup we shall introduce the notions of the variation and of the semivariation for semigroup valued additive set functions. We shall prove that these notions have the usual properties.

By the Diagonal Theorem from |4| we shall prove, in quite an elementary way, two theorems on the uniform boundedness of a family of regular additive exhaustive set functions on a Borel algebra (i.e. on a σ-algebra generated by the open subsets of a compact Hausdorff topological space). Namely, if the domain is not a σ-algebra, the Nikodym boundedness theorem, in general, is not true (see |2|). But there are known uniform boundedness theorems in which the initial boundedness conditions are given on some subfamilies which are not σ-algebras (for example in |5|). In the stated theorems this subfamily is the family of open sets.

2. THE VARIATION AND SEMIVARIATION

Let X be a commutative semigroup with the neutral element 0 endowed with a triangle functional f such that

$$f(x+y) \leq f(x) + f(y),$$

$$f(x+y) \geq f(x) - f(y) \quad \text{and}$$

$$f(0) = 0 \qquad \text{(see |4|).}$$

REMARK 1.

The topology of a uniform semigroup is generated by a family of special pseudometrics - H.Weber |7|. If follows from this construction, that there exists a family of triangle functions on every uniform semigroup - E.Pap |6|.

Let Σ be an algebra of subsets of the set S. A set function μ defined on the algebra Σ with the values in X is <u>additive</u> if whenever E_1 and E_2 are disjoint members of Σ then

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

We suppose that there exists $B \in \Sigma$ such that $\mu(B) = 0$. It follows from additivity, by the preceding assumption, that $\mu(\emptyset) = 0$.

A set function $\mu : \Sigma \to X$ is said to be <u>exhaustive</u> (f-exhaustive) if $\lim_{n\to\infty} f(\mu(E_n)) = 0$ for every sequence (E_n) of disjoint members of Σ.

The variation of an additive set function $\mu : \Sigma \to X$ is the nonnegative set function $|\mu|$ defined for $E \in \Sigma$ by

$$|\mu|(E) = \sup_{\pi} \sum_{A\in\pi} f(\mu(A))$$

where the supremum is taken over all partitions π of E into a finite number of pairwise disjoint members of Σ.

Obviously $f(\mu(E)) \leq |\mu|(E)$ for each $E \in \Sigma$ and $|\mu|(\emptyset) = 0$.

PROPOSITION 1. *If* $\mu : \Sigma \to X$ *is an additive set function on an algebra* Σ, *then* $|\mu|$ *is also additive on* Σ.

P r o o f. Let E_1 and E_2 be two disjoint sets from Σ. If $|\mu|(E_1 \cup E_2) < \infty$, then for $\varepsilon > 0$ we can choose finite systems $\{A_i^1\}$ and $\{A_i^2\}$ of pairwise disjoint sets from Σ such that $A_i^1 \subset E_1$ and $A_i^2 \subset E_2$ and

$$|\mu|(E_i) \leq \sum_j f(\mu(A_j^i)) + \varepsilon \quad \text{for} \quad i = 1,2.$$

Then we have

$$|\mu|(E_1) + |\mu|(E_2) \leq \sum_j f(\mu(A_j^1)) + \sum_j f(\mu(A_j^2)) + 2\varepsilon \leq$$

$$\leq |\mu|(E_1 \cup E_2) + 2\varepsilon .$$

Since $\varepsilon > 0$ was arbitrary we obtain the finite superadditivity of $|\mu|$, i.e.

$$(1) \qquad |\mu|(E_1) + |\mu|(E_2) \leq |\mu|(E_1 \cup E_2) .$$

On the other hand, we take partition $\{A_j\}$ of $E_1 \cup E_2$ into a finite number of pairwise disjoint members of Σ. We take $A_j^1 = E_1 \cap A_j$ and $A_j^2 = E_2 \cap A_j$. Then we have

$$\sum_j f(\mu(A_j)) \leq \sum_j f(\mu(A_j^1)) + \sum_j f(\mu(A_j^2) \leq |\mu|(E_1) + |\mu|(E_2) .$$

Hence

$$|\mu|(E_1 \cup E_2) \leq |\mu|(E_1) + |\mu|(E_2) .$$

By the preceding inequality and (1) we obtain the additivity of $|\mu|$. The last inequality implies also the additivity in the case $|\mu|(E_1 \cup E_2) = \infty$.

REMARK 2.

$|\mu|$ is superadditive in general, i.e.

$$|\mu|(\bigcup_{i\in I} E_i) \geq \sum_{i\in I} |\mu| (E_i) .$$

This follows in the same way as in |3| , p.35.

An additive set function $\mu : \Sigma \to X$ is said to be of bounded variation if $|\mu|(S) < \infty$.

If μ is of bounded variation then it is also bounded (f-bounded), but in general the converse is not true (for example $X = L^{\infty}[0,1]$, Σ is a σ-algebra of Lebesgue measurable subsets of $[0,1]$ and $\mu(E) = \chi_E$).

The semivariation of an additive set function $\mu : \Sigma \to X$ is the nonnegative set function $\|\mu\|$ defined for $E \in \Sigma$ by

$$\|\mu\|(E) = \sup\{f(\mu(H)) \mid H \in \Sigma,\ H \subset E\}.$$

Obviously:

(i) $\|\mu\|(\emptyset) = 0$ (ii) $f(\mu(E)) \leq \|\mu\|(E)$ for each $E \in \Sigma$
(iii) $\|\mu\|(E) \leq |\mu|(B)$ for each $E \in \Sigma$, (iv) $\|\mu\|(F) \leq \|\mu\|(E)$ for $F \subset E$ and $F, E \in \Sigma$.

PROPOSITION 2. *If $\mu : \Sigma \to X$ is an additive set function on an algebra Σ, then $\|\mu\|$ is subadditive on Σ, i.e.*

$$\|\mu\|(E_1 \cup E_2) \leq \|\mu\|(E_1) + \|\mu\|(E_2)$$

for every $E_1, E_2 \in \Sigma$ such that $E_1 \cap E_2 = \emptyset$.

P r o o f. Let E_1 and E_2 be two disjoint sets from Σ. Let H be an arbitrary subset of $E_1 \cup E_2$. Then we have

$$f(\mu(H)) = f(\mu(H \setminus E_1 \cup H \setminus E_2)) \leq f(\mu(H \setminus E_1)) +$$
$$+ f(\mu(H \setminus E_2)) \leq \|\mu\|(E_1) + \|\mu\|(E_2).$$

Hence the desired inequality follows.

An additive set function $\mu : \Sigma \to X$ is said to be of bounded semivariation if $\|\mu\|(S) < \infty$. By the definition we obtain directly that the semivariation of the additive set function $\mu : \Sigma \to X$ is bounded iff μ is bounded.

REMARK 3.

The semivariation is usually defined for a Banach space valued additive set function as

$$\sup\{|x^*\mu|(E)\,|\,x^* \in X^*\,,\ \|x^*\| \leq 1\}$$

where $|x^*\mu|$ is the variation of $x^*\mu$ (see |2|), or, equivalently as

$$\sup\{\|\sum_{A_m \in \pi} \varepsilon_n \mu(A_n)\|\}$$

where the supremum is taken over all partitions π of E into finitely many disjoint members of Σ and over all finite collections $\{\varepsilon_n\}$ satisfyng $|\varepsilon_n| \leq 1$ (|3|, p.51 or Proposition 11.a. from |2|, p.4. Our definition of the semivariation, in the case of Banach space valued additive set functions, is equivalent, in the sense of the norm, to the usual one (Proposition 11.b. p.4. from |2|).

PROPOSITION 3. *An additive set function* $\mu : \Sigma \to X$ *is exhaustive iff* $\lim_{n\to\infty} \|\mu\|(E_n) = 0$ *for every sequence* (E_n) *of disjoint members of* Σ.

P r o o f. Let μ be an exhaustive additive set function. Suppose that $\|\mu\|$ does not satisfy the proposition. Then there exist $\delta > 0$, a sequence (E_n) of disjoint members of Σ for which

$$\|\mu\|(E_n) \geq \delta$$

holds. For each n and ε such that $\delta > \varepsilon > 0$ there is $H_n \in \Sigma$ such that $H_n \subset E_n$ and

$$\|\mu\|\ (E_n) \leq f(\mu(H_n)) + \varepsilon .$$

Then the sequence (H_n) consists of disjoint members of Σ such that

$$f(\mu(H_n)) \geq \delta - \varepsilon > 0$$

holds for each n, which contradict the exhaustivity of μ. The converse implication follows from the inequality

$$f(\mu(E)) \leq \|\mu\|(E) \quad \text{for each } E \in \Sigma.$$

THEOREM 1. *An exhaustive additive set function* $\mu:\Sigma\to X$ *on an algebra* Σ *of subsets of* S *is of bounded semivariation.*

P r o o f. Let us suppose that the theorem is not true. Then there exists $H_1 \in \Sigma$ such that

$$(2) \qquad f(\mu(H_1)) \geq 1 + 2f(\mu(S))$$

Since

$$f(\mu(H_1)) - f(\mu(S\setminus H_1)) \leq f(\mu(H_1) + \mu(S\setminus H_1)) = f(\mu(S)) ,$$

we obtain

$$f(\mu(S\setminus H_1)) \geq 1 .$$

Since

$$\|\mu\|(S) \leq \|\mu\|(H_1) + \|\mu\|(S\setminus H_1) ,$$

either $\|\mu\|(H_1)$ or $\|\mu\|(S\setminus H_1)$ is infinite. If H_1 is with unbounded semivariation we put $E_1 = H_1$, otherwise we take $E_1 = S\setminus H_1$ we have that E_1 has the unbounded semivariation and $f(\mu(E_1)) \geq 1$. Now we take E_1 instead of S and repeat the preceding procedure.

We obtain $E_2 \in \Sigma$ and $E_2 \subset E_1$ such that the semivariation of E_2 is unbounded and $f(\mu(E_2)) \geq 2$. By iterating we construct a nonincreasing sequence (E_n) from Σ such that the semivariation of E_n is unbounded for each n and

$$(3) \qquad f(\mu(E_n)) \geq n .$$

Let $F_n = E_n\setminus E_{n+1}$. Since $\mu(E_n\setminus E_{n+1}) + \mu(E_{n+1}) = \mu(E_n)$ we obtain

$$f(\mu(F_n)) \geq f(\mu(E_{n+1})) - f(\mu(E_n)).$$

Then, by (3), it follows that $f(\mu(F_n)) \geq 1$ for some disjoint subsequence of (F_n) from Σ, which contradicts the exhaustive of μ.

3. UNIFORM BOUNDEDNESS ON BOREL SETS

In this section Σ is a σ-algebra of subsets of a set S. An additive set function $\mu : \mathcal{B} \to X$ defined on the collection $\mathcal{B}$ of Borel sets of a compact Hausdorff topological space T is regular on a Borel set B if for every $\varepsilon > 0$ there exists a compact set $K \subset B$ and an open set $O \supset B$ such that

$$\|\mu\|(O \setminus K) < \varepsilon .$$

A set function $\mu : \Sigma \to X$ is <u>f-superadditive</u> if for every sequence (E_n) of disjoint members of a σ-algebra Σ

$$\lim_{n\to\infty} f(\mu(\bigcup_{i=1}^{n} E_i)) \leq f(\mu(\bigcup_{i=1}^{\infty} E_i))$$

holds. We shall use the following.

DIAGONAL THEOREM. ([4]). *If* $x_{ij} \in X$ $(i,j \in N)$ *and*

$$\lim_{j\to\infty} f(x_{ij}) = 0 \quad \textit{for} \quad i=1,2,\ldots ,$$

then there exist an infinite set I *of positive integers and a subset* J *(finite or infinite) of* I *such that, for all* $i \in I$, *we have*

$$\sum_{j\in J} f(x_{ij}) < \infty$$

$$f(\sum_{j\in J} x_{ij}) \geq \frac{1}{2} f(x_{ii}) .$$

(where J *is infinite)*

$$f(\sum_{j\in J} x_{ij}) \overset{\text{def}}{=} \lim_{n\to\infty} f(\sum_{s=1}^{n} x_{ij_s}),$$

(j_s) *is the increasing sequence of all elements of* J.

THEOREM 2. *Let* $(\mu_\alpha)_{\alpha\in A}$ *be a family of* X*-valued regular additive and f-superadditive set functions defined on the Borel subsets of a compact Hausdorff topological space* T. *If*

$$\{\mu_\alpha(O) \mid \alpha \in A\}$$

is bounded on every open set O, *then the family* $(\mu_\alpha)_{\alpha\in A}$ *is uniformly bounded, i.e.*

$$\sup\{f(\mu_\alpha(B)) \mid \alpha \in A,\ B \in \mathcal{B}\} < \infty \quad .$$

P r o o f. Since the general case can be reduced to the proof for the sequence, we take a sequence (μ_n) instead of the family.

To prove the uniform boundedeness of (μ_n), it suffices to prove that every point in T belongs to an open set O on which

$$\sup_{n,A} \{f(\mu_n(A)),\ n \in N,\ A \subset O\} < \infty$$

Suppose that this is not true. Then there exists $x \in T$ such that $\sup_{n\in N} \|\mu_n\|(O)$ is not bounded for every open set O such that $x \in O$. Let O be an open set which contains x. Then there exists a Borel set $B \subset O$ and $n_1 \in N$ such that

$$f(\mu_{n_1}(B)) > 4 + 2 \sup_n f(\mu_n(\{x\})) \quad . \tag{4}$$

Since μ_{n_1} is regular, there exists a compact set $K_o \subset B$ such that

$$\|\mu_{n_1}\|(O \setminus K_o) < 1. \tag{5}$$

Then, by the inequalities

$$f(\mu_{n_1}(K_o)) + f(\mu_{n_1}(B \quad K_o)) \geq f(\mu_{n_1}(B)),$$
$$f(\mu_{n_1}(B \setminus K_o)) \leq \|\mu_{n_1}\|(B \setminus K_o) \leq \|\mu_{n_1}\|(O \setminus K_o),$$

(4) and (5) we obtain

$$f(\mu_{n_1}(K_o)) > 3 + 2 \sup_n f(\mu_n(\{x\})) \quad .$$

Let $K_1 = K_o \cup \{x\}$. Then we have

$$f(\mu_{n_1}(K_1)) > 3 + \sup_n f(\mu_n(\{x\})) \quad .$$

By the regularity of μ_{n_1} there exists an open set U such that $O \supset U \supset K_1$ and

$$\|\mu_{n_1}\| (U \setminus K_1) < 1 .$$

Hence, by the inequality

$$f(\mu_{n_1}(U)) \geq f(\mu_{n_1}(K_1)) - f(\mu_{n_1}(U \setminus K_1)),$$

we obtain

$$(6) \qquad f(\mu_{n_1}(U)) > 2 + \sup_n f(\mu_n(\{x\})).$$

We find, again by regularity, an open set W such that $\{x\} \subset W \subset U$ and $\|\mu_{n_1}\| (W \setminus \{x\}) < 1$. Let H be an open set such that $x \in H \subset \bar{H} \subset W$. Then we have

$$(7) \qquad f(\mu_{n_1}(\bar{H})) \leq \|\mu_{n_1}\| (\bar{H} \setminus \{x\}) + f(\mu_{n_1}(\{x\})) \leq$$

$$\leq \|\mu_{n_1}\| (W \setminus \{x\}) + f(\mu_{n_1}(\{x\}) < 1 + \sup_n f(\mu_n(\{x\})).$$ If we

take that $E_1 = U \setminus \bar{H}$, then we have $E_1 \subset 0$, $E_1 \cap H = \emptyset$. By the inequality

$$f(\mu_{n_1}(E_1)) + f(\mu_{n_1}(\bar{H})) \geq f(\mu_{n_1}(U)) ,$$

(6) and (7), we obtain

$$f(\mu_{n_1}(E_1)) > 1.$$

Using the same procedure, the fact that $x \in H$ and that the sequence (μ_n) is unbounded on H, and taking in the inequality (4) $5+2\sup_n f(\mu_n(\{x\}))$ instead of $4+2 \sup_n f(\mu_n(\{x\}))$, we obtain an integer $n_2 > n_1$, open sets $E_2, H_1 \subset H$ such that $E_1 \cap E_2 = \emptyset$, $E_2 \cap H_1 = \emptyset$, $x \in H_1$ and $f(\mu_{n_2}(E_2)) > 2$. Continuing, we obtain a sequence of positive integers (n_k) and a sequence of pairwise disjoint open sets (E_k) such that

$$(8) \qquad f(\mu_{n_k}(E_k)) > k \quad \text{for every } k \in N.$$

Now we take $x_{ij} = \mu_{n_i}(E_j)$ for $i,j \in N$. Then we have for each $i \in N$

$$\lim_{j\to\infty} f(x_{ij}) = 0 .$$

Then there exist , by the Diagonal Theorem, an infinite set $I \subset N$ and its subset J such that

$$f\Big(\sum_{j\in J} x_{ij}\Big) \geq \frac{1}{2} f(x_{ii}) \tag{9}$$

for each $i \in I$. Since μ_{n_i} are additive and f-superadditive we have

$$\lim_{n\to\infty} f\Big(\sum_{s=1}^{n} \mu_{n_i}(E_{j_s})\Big) = \lim_{n\to\infty} f\Big(\mu_{n_i}\Big(\bigcup_{s=1}^{n} E_{j_s}\Big)\Big) \leq f\Big(\mu_{n_i}\Big(\bigcup_{s=1}^{\infty} E_{j_s}\Big)\Big)$$

(if J is finite we need only the first equality). Then by (9) and (8) we obtain

$$f\Big(\mu_{n_i}\Big(\bigcup_{j\in J} E_j\Big)\Big) > \frac{i}{2}$$

for each $i \in I$, which is a contradiction with the boundedness of (μ_n) on every open set.

Using the main part of the proof of Theorem 2 we obtain a uniform boundedness - type theorem connected with the variation.

THEOREM 3. *Let* $(\mu_\alpha)_{\alpha\in A}$ *be a family of* X*-valued regular additive set functions defined on the Borel subsets of a compact Hausdorff topological space* T. *If* μ_α *is of bounded variation on every open set, for each* $\alpha \in A$, *then the family is uniformly bounded.*

P r o o f. The proof is verbatum the same as up to (9). If J is infinite, we have

$$\lim_{n\to\infty} f\Big(\sum_{s=1}^{n} \mu_{n_i}(E_{j_s})\Big) \leq \lim_{n\to\infty} \sum_{s=1}^{n} f(\mu_{n_i}(E_{j_s})) \leq$$

$$\leq \lim_{n\to\infty} \sum_{s=1}^{n} |\mu_{n_i}|(E_{j_s}) \leq |\mu_{n_i}|\Big(\bigcup_{j\in J} E_j\Big)$$

where (j_s) is the increasing sequence of all elements of J. By (9) and (8) we have

$$|\mu_{n_i}|\Big(\bigcup_{j\in J} E_j\Big) > \frac{i}{2}$$

for every $i \in I$ which is a contradiction.

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REZIME

O ADITIVNOJ EKSHAUSTIVNOJ FUNKCIJI SKUPA SA VREDNOSTIMA U POLUGRUPI

U radu se ispituje funkcija skupa μ sa vrednostima u komutativnoj polugrupi X sa neutralnim elementom 0 i trougaonom funkcionelom f, za koju važi: $f(x+y) \leq f(x) + f(y)$, $f(x+y) \geq f(x) - f(y)$ i $f(0) = 0$. Izdvajaju se aditivne i ekshaustivne funkcije skupa na algebri Σ skupova. Uvode se ne-negativne funkcije skupa, pridružene funkciji skupa sa vrednostima u polugrupi μ, varijacija $|\mu|$ i poluvarijacija $\|\mu\|$.

Pomoću Dijagonalne teoreme iz |4|, dokazuju se dve teoteme o uniformnoj ograničenosti:

Teorema 2. Neka je $(\mu_\alpha)_{\alpha\in A}$ familija X vrednosnih regularnih aditivnih i f-superaditivnih funkcija skupa definisanih na Borelovim podskupovima kompaktnog Hausdorfovog topološkog prostora T. Ako je $\{\mu_\alpha(O) \mid \alpha \in A\}$ ograničeno na svakom otvorenom skupu O,tada je familija $(\mu_\alpha)_{\alpha\in A}$ i uniformno ograničena na svim Borelovim skupovima.

Teorema 3. Neka je $(\mu_\alpha)_{\alpha \in A}$ familija X vrednosnih regularnih aditivnih funkcija skupa definisanih na Borelovim podskupovima kompaktnog Hausdorfovog topološkog prostora T. Ako je μ_α ograničene varijacije na svakom otvorenom skupu, za svako $\alpha \in A$, tada je familija uniformno ograničena.

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UNBOUNDED SOLUTIONS OF A NONLINEAR DIFFERENTIAL EQUATION

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Consider an initial value problem

(1) $$y'' = f(x)y^{\lambda} \quad ; \quad y(a) = b_o \ , \quad y(a) = b_1$$

where $\lambda > 1$, $a > 0$, and $f(x)$ is continuous and positive for $x > a$.

By an unbounded solution of (1) is meant a solution possessing the property

$$y(x) \to \infty \quad \text{for} \quad x \to \omega - 0$$

where ω is some positive real number.

Such solutions were first considered by R.H.Fowler in 1931, |1| , for $f(x) = x^r$, r real.

J.Karamata and V.Marić |2| proved the existence of unbounded solutions of (1) for any $b_o > 0$, $b_1 > 0$ provided that

(2) $$f(x) \geq m > 0 \ , \quad x > 0 \ .$$

If (2) holds for $a \leq x \leq b$ only, then the above result is still valid but only for some appropriately chosen values of b_1.

The interest for unbounded solutions has been renewed by E.Hille (cf. |3|, |4|, |5|) who proved the existence of these for the Thomas-Fermi ($f(x) = x^{-1/2}$; $\lambda = 1{,}5$) and Emden ($f(x) = x^{1-m}$, $\lambda = m$, $m > 1$) equations occurring in various applications, with $b_o \geq 0$, $b_1 \geq 0$ and $b_o + b_1 > 0$.

Hille's result is generalized by L.E.Bobisud |6| for the equation

$$(3) \qquad y'' = f(x)g(y)$$

where f,g are continuous and positive for $x>0$, $y>0$; if

$$(4) \qquad \int_a^\infty \left(\int_a^x g(u)du\right)^{-1/2} dx < \infty$$

and

$$f'(x)f^{-3/2}(x) \to 0 \quad \text{for} \quad x \to \infty .$$

A special case $b_1 = 0$ ($b_0 > 0$) is studied by S.B.Eliason |7| .

V.Marić and M.Skendžić |8| proved an existence theorem for unbounded solutions of the equation (3) for $b_0 > 0$, $b_1 \geq 0$ and under less restrictive conditions on f(x):

$$f(x) \geq h(x) \ , \qquad \int^\infty \sqrt{h(x)}dx = \infty$$

where h(x) is a positive, continuous, decreasing function which tends to zero for $x \to \infty$; g(u) satisfies (4).

In the present we shall prove the existence of unbounded solutions for $y'(a) = b_1 < 0$. To do this we shall first prove two lemmas.

LEMMA 1. *Let* y(x), Y(x) *be non-negative solutions of the initial value problems*

$$y'' = yf(x,y) \ , \qquad y(a) = Y(a)$$

$$Y'' = YF(x,Y) \ , \qquad y'(a) < Y'(a)$$

where f,F *are continuous functions such that* $0 < f(u,v) < F(u,v)$ *for* $a \leq u \leq b$ *and* $v > 0$, *and let* F(u,v) *be a strictly increasing function of* v *for each* u.

Then $y(x) < Y(x)$ *and* $y'(x) < Y'(x)$ *for* $a < x \leq b$.

P r o o f. For $a \leq x \leq b$ we have

$$y(x) = y(a)+(x-a)y'(a) + \int_a^x (x-s)y(s)f(s,y(s))ds$$

$$y(x) = y(a) + (x-a)y'(a) + \int_a^x (x-s)y(s)f(s,y(s))ds$$

$$Y(x) = Y(a) + (x-a)Y'(a) + \int_a^x (x-s)Y(s)F(s,Y(s))ds$$

So that

$$(5)\quad Y(x)-y(x)=(x-a)(Y'(a)-y'(a)) + \int_a^x (x-s)(Y(s)F(s,Y(s)) - y(s)F(s,y(s)))ds .$$

Since $y'(a) < Y'(a)$ there exists $\varepsilon > 0$ such that $Y(x)-y(x) > 0$ for $a < x < a + \varepsilon$. Suppose $Y(x) \leq y(x)$ for some $x \in (a,b]$. Then there exists $c \in (a,b]$ such that $Y(x)-y(x) > 0$ for $a < x < c$ and $Y(c)=y(c)$. Since $y(x)$ and $Y(x)$ are non-negative, $Y(x)F(x,Y(x))-y(x)f(x,y(x)) > 0$ for $a < x < c$. Putting $x=c$, in (5) the left hand side is zero and the right hand side is positive, which is a contradiction. Thus $y(x) < Y(x)$ for $a < x \leq b$. Moreover

$$y'(x) = y'(a) + \int_a^x y(s)f(s,y(s))ds$$

$$Y'(x) = Y'(a) + \int_a^x Y(s)F(s,Y(s))ds.$$

Hence $y'(x) < Y'(x)$ for $a \leq x \leq b$.

LEMMA 2. *Suppose* $y_1(x)$ *and* $y_2(x)$ *are two positive solutions of*

$$y" = yF(x,y) \;;\quad y_1(a) = y_2(a);\quad y_1'(a) < y_2'(a)$$

where F *satisfies the same conditions as in Lemma 1. Then* $y_2(x) - y_1(x) \geq (x-a)(y_2'(a)-y_1'(a))$ *for* $x \geq a$.

P r o o f. According to Lemma 1. $y_2(x) = y_1(x)$. Therefore

$$y_2''(x)-y_1''(x) = y_2(x)F(x,y_2(x)) - y_1(x)F(x,y_1(x)) \geq 0.$$

So $y_2'(x)-y_1'(x) \geq y_2'(a)-y_1'(a)$ and hence, after integration over $[a,x]$, the proof is finished.

The consequence of this Lemma is that the positive, decreasing solution defined for all $x > a$ of the initial value problem

$$y" = yF(x,y) \;;\quad y(a) = b_o \;;\quad y'(a) = b_1 < 0$$

(provided that it exists) is unique and all other are not bounded (including both i.e. such that $y(x) \to \infty$, $x \to \omega - 0$, or $y(x) \to \infty$, $x \to \infty$).

The existence of such solutions is proved by P.K.Wong |9| as follows: there exists a positive decreasing solution $y(x)$ which tends, for $x \to \infty$, to a positive constant iff there is a $\beta > 0$ such that $\int^{\infty} xF(x,\beta)dx$ converges. The divergence of that integral for all $\beta > 0$ is, on the other hand, a neccessary and sufficient condition for the existence of a solution tending to zero for $x \to \infty$.

When $y'(a) > b_1$, corresponding solutions, if they are defined for all $x \geq a$, are not slower than a linear function. By some restriction on F, existence of solutions such that $y \sim kx$, $x \to \infty$ was proved by P.Waltman |10|.

Denote by $y(x)$ the positive solutions of the initial value problems

$$(6) \qquad y'' = yF(x,y) \quad ; \quad y(a) = b_0 \quad ; \quad y'(a) = b_1$$

and let b_∞ stand for the initial slope of the unique positive decreasing bounded solution of (6) (defined for all $x \geq a$). Then we prove the following.

THEOREM *If the initial value problem*

$$Y'' = YF(x,Y) \quad ; \quad Y(x_0) = b_0 \quad ; \quad Y'(x_0) = 0$$

has unbounded solutions for each x_0, *then any positive solution* $y(x)$ *of (6), satisfying* $y'(a) > b_\infty$, *is unbounded too.*

P r o o f. First take $b_\infty < b_1 < 0$, then according to Lemma 2 for the solution $y(x)$ there exists a point $x_0 > a$ such that $y(x_0) = y(a) = b_0$. Since the solution is convex, $y'(x_0) > Y'(x_0) = 0$ and then, by Lemma 1, is unbounded.

If $b_1 \geq 0$, one may take $x_0 = a$ and repeat the argument.

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REZIME

NEOGRANIČENA REŠENJA NELINEARNE DIFERENCIJALNE JEDNAČINE

U ovom radu je pokazano kada početni problem

$$y'' = yF(x,y)\ , \ y(a) = b_o, \quad y'(a) = b_1$$

ima neograničeno rešenje za $b_1 < 0$.

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A COMMON FIXED POINT THEOREM OF A FAMILY OF MAPPINGS IN PROBABILISTIC LOCALLY CONVEX SPACES

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In this paper a theorem which gives the necessary and sufficient condition for the existence of a unique fixed point for the mappings S,T and A (defined in Theorem 2) is proved.

First we shall give some definitions and notations which we shall use later.

DEFINITION 1 |1| . *Let* X *be a linear space over the real or complex field* K *and for every* i *in the index set* I *there is a function* $F^i : X \to \Delta^+$, *where* Δ^+ *is the family of distribution functions* F *such that* $F(0) = 0$. *We shall denote* $F^i(p)$ *by* F^i_p $(i \in I,\ p \in X)$.

$(X, \{F^i\}_{i\in I}, t)$ *is called a locally convex probabilistic space (LCP-space) if and only if for each* $i \in I$ *the following conditions are satisfied:*

1. $F^i_o = H$, *where* $H(\varepsilon) = \begin{cases} 0, & \varepsilon \le 0 \\ 1, & \varepsilon > 0 \end{cases}$,
2. $F^i_{\lambda p}(\varepsilon) = F^i_p\left(\dfrac{\varepsilon}{|\lambda|}\right)$ *for every* $\lambda \in K,\ p \in X,\ \varepsilon > 0, (\lambda \neq 0)$
3. $F^i_{p+q}(\varepsilon_1+\varepsilon_2) \geq t(F^i_p(\varepsilon_1),\ F^i_q(\varepsilon_2))$ *for every* $p, q \in X$,

$\varepsilon_1 > 0,\ \varepsilon_2 > 0$, *where the mapping* $t : [0,1] \times [0,1] \to [0,1]$ *is a T-norm* |2| .

The (ε,λ)- topology is introduced by the following definition.

DEFINITION 2 |1|. *A net* $\{p_d\}_{d\in D}$ *converges to* o *if and only if for each* $i \in I$, *every* $\varepsilon > 0$ *and every* $\lambda \in (0,1)$ *there exists* $d_o \in D$ *such that*

$$F^i_{p_d}(\varepsilon) \geq 1-\lambda \quad \textit{if} \quad d \geq d_o .$$

In what follows we suppose that

$$F^i_p = H \quad \text{for each } i \in I \text{ if and only if } \quad p = o.$$

In a similar way one can introduce the notion of a Cauchy sequence and the notion of completeness.

Some fixed point theorems in probabilistic spaces are proved in |1|, |2|.

THEOREM 1 |3|. *Let* S *and* T *be continuous mappings of a complete normed space* $(X, \|\cdot\|)$ *into istelf. Then* S *and* T *have a common fixed point in* X *if and only if there exists a continuous mapping* A *of* X *into* $SX \cap TX$ *which commutes with* S *and* T *and satisfies the inequality*

$$\|Ax-Ay\| \leq \alpha \, \|Sx-Ty\|$$

for every $x,y \in X$, *where* $0 < \alpha < 1$. S ,T *and* A *then have a unique common fixed point.*

THEOREM 2. $(X,\{F^i\}_{i\in I},t)$ *be a sequentially complete probabilitic locally convex space with continuous* T-*norm* t *and* S *and* T *be continuous mappings of* X *into* X.

The mappings S *and* T *have a unique common fixed point in* X *if and only if there exists a continuous mapping* $A : X \to SX \cap TX$ *which commutes with* S *and* T *so that* AX *is a probabilistic bounded subset of* X *and satisfies the following conditions:*

1. *For every* $i \in I$ *there exists* $q(i) > 0$ *and* $f(i) \in I$ *such that*

$$F^{i}_{Ax-Ay}(\varepsilon) \geq F^{f(i)}_{Tx-Sy}\left(\frac{\varepsilon}{q(i)} \right)$$

for every $\varepsilon > 0$ *and every* $x, y \in X$.

2. *For every* $i \in I$ *there exist numbers* $n_i \in N$ *and* $Q(i) \in (0,1)$ *such that*

$$q(f^n(i)) \leq Q(i) < 1$$

for every $n > n_i$.

3. *For every* $i \in I$ *there exists* $g(i) \in I$ *such that*

$$F^{f^n(i)}_{x}(\varepsilon) \geq F^{g(i)}_{x}(\varepsilon)$$

for every $\varepsilon > 0$, *every* $x \in X$ *and every* $n \in N$.

Then there exists one and only one element $x^* \in X$ *such that* Ax^* *is the unique common fixed point for the mappings* A, S *and* T .

P r o o f. First, let us prove the necessity of the conditions 1, 2 and 3. Let $z \in X$ be such an element that

$$z = Az = Sz = Tz \ .$$

The mapping A is defined by $Ax = z$ for all $x \in X$.

The mapping A commutes with S and T since

$$A(Sx) = Ay = z \ , \qquad S(Ax) = Sz = z$$

and

$$A(Tx) = Av = z, \qquad T(Ax) = Tz = z.$$

Condition 1 will be satisfied because

$$F^{i}_{Ax-Ay}(\varepsilon) = F^{i}_{z-z}(\varepsilon) = 1 \geq F^{i}_{Tx-Sy}\left(\frac{\varepsilon}{q(i)} \right)$$

for all $x, y \in X$, i.e. $f(i) = i$ for all $i \in I$ and $q(i)$ is any element from $(0,1)$.

Since $f(i) = i$, contition 3 is also satisfied, i.e.

$$F^{f^n(i)}_{x}(\varepsilon) = F^{i}_{x}(\varepsilon)$$

for all $x \in X$ and all $\varepsilon > 0$. We have that $g(i) = i$, $i \in I$.

Now, we shall prove that conditions 1,2 and 3 are sufficient. We form a sequence $\{x_n\}_{n \in N \cup \{0\}}$ such that

$$Ax_{2n-2} = Sx_{2n-1}\,, \qquad Ax_{2n-1} = Tx_{2n}$$

for all $n \in N$. Such a sequence always exists because $AX \subset TX \cap SX$.

First, we prove that the sequence $\{Ax_n\}_{n \in N \cup \{0\}}$ is a Cauchy sequence. It is easy to see that

$$F^{i}_{Ax_{2n}-Ax_{2n-1}}(\varepsilon) \geq F^{f(i)}_{Tx_{2n}-Sx_{2n-1}}\left(\frac{\varepsilon}{q(i)}\right) =$$

$$= F^{f(i)}_{Ax_{2n-1}-Ax_{2n-2}}\left(\frac{\varepsilon}{q(i)}\right) \geq F^{f^2(i)}_{Tx_{2n-2}-Sx_{2n-1}}\left(\frac{\varepsilon}{q(i)q(f(i))}\right) \geq$$

$$\geq F^{f^2(i)}_{Ax_{2n-2}-Ax_{2n-3}}\left(\frac{\varepsilon}{q(i)q(f(i))}\right) \geq \dots \geq$$

$$\geq F^{f^{2n-1}(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon}{\prod_{s=o}^{2n-2} q(f^s(i))}\right)$$

for all $\varepsilon > 0$, $i \in I$, $n \in N$. Also we have that

$$F^{i}_{Ax_{2n+1}-Ax_{2n}}(\varepsilon) = F^{i}_{Ax_{2n}-Ax_{2n+1}}(\varepsilon) \geq \dots \geq$$

$$\geq F^{f^{2n}(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon}{\prod_{s=o}^{2n-1} q(f^s(i))}\right)$$

for all $\varepsilon > 0$, $i \in I$, $n \in N$. So for every $n \in N$ we have

$$F^{i}_{Ax_n-Ax_{n-1}}(\varepsilon) \geq F^{f^{n-1}(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon}{\prod_{s=o}^{n-2} q(f^s(i))}\right)$$

for all $\varepsilon > 0$, $i \in I$.

We see that for all $m > k$

$$F^{i}_{Ax_{2k}-Ax_{2m+1}}(\varepsilon) \geq F^{f(i)}_{Sx_{2m+1}-Tx_{2k}}\left(\frac{\varepsilon}{q(i)}\right) =$$

$$= F^{f(i)}_{Ax_{2m}-Ax_{2k-1}}\left(\frac{\varepsilon}{q(i)}\right) \geq \dots \geq F^{f^{2k}(i)}_{Ax_o-Ax_{2m+1-2k}}\left(\frac{\varepsilon}{\prod_{s=o}^{2k-1} q(f^s(i))}\right).$$

If $2k > 2m+1$ then we have

$$F^{i}_{Ax_{2k}-Ax_{2m+1}}(\varepsilon) \geq F^{f(i)}_{Ax_{2k-1}-Ax_{2m}}\left(\frac{\varepsilon}{q(i)}\right) \geq \dots \geq$$

$$\geq F^{f^{2m+1}(i)}_{Ax_{2k-2m-1}-Ax_o}\left(\frac{\varepsilon}{\prod_{s=o}^{2m} q(f^s(i))}\right)$$

for every $\varepsilon > 0$.

From the last two inequalities we can prove easily that for $n = 2k$, $p = 2m+1$ or for $n = 2k+1$, $p = 2m+1$ the following is satisfied

$$F^{i}_{Ax_{n+p}-Ax}(\varepsilon) \geq F^{f^n(i)}_{Ax_o-Ax_p}\left(\frac{\varepsilon}{\prod_{s=o}^{n-1} q(f^s(i))}\right).$$

When $n = 2k$, $p = 2m$ or $n = 2k+1$, $p = 2m$ we have

$$F^{i}_{Ax_{n+p}-Ax_n}(\varepsilon) = F^{i}_{Ax_{n+p}-Ax_{n+1}+Ax_{n+1}-Ax_n}\left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) \geq$$

$$\geq t\left(F^{i}_{Ax_{n+p}-Ax_{n+1}}\left(\frac{\varepsilon}{2}\right), F^{i}_{Ax_{n+1}-Ax_n}\left(\frac{\varepsilon}{2}\right)\right) \geq \dots \geq$$

$$\geq t\left(F^{f^{n+1}(i)}_{Ax_{p-1}-Ax_o}\left(\frac{\varepsilon}{2\prod_{s=o}^{n} q(f^s(i))}\right), F^{f^n(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon}{2\prod_{s=o}^{n-1} q(f^s(i))}\right)\right).$$

Now we shall prove that for every $\varepsilon > 0$, $\lambda \in (0,1)$ and $i \in I$ there exists $N(i,\varepsilon,\lambda)$ such that

$$F^i_{Ax_{n+p}-Ax_n}(\varepsilon) > 1 - \lambda$$

for every $n \geq N(i,\varepsilon,\lambda)$ and $p \in N$.

The set AX is probabilistic bounded which means that

$$\sup_{\varepsilon} D^i_{AX}(\varepsilon) = 1$$

for all $i \in I$ and so for each $\lambda \in (0,1)$, there exists $\varepsilon_i(\lambda) > 0$ such that

$$D^i_{AX}(\varepsilon_i(\lambda)) > 1 - \lambda.$$

Since

$$D^i_{AX}(\varepsilon_i) = \sup_{\delta<\varepsilon_i} \inf_{u,v\in AX} F^i_{u-v}(\delta)$$

it follows that:

$$\sup_{\delta<\varepsilon_i} \inf_{u,v\in AX} F^i_{u-v}(\delta) > 1 - \lambda,$$

and we have that

$$\inf_{u,v\in AX} F^i_{u-v}(\varepsilon_i(\lambda)) \geq \inf_{u,v\in AX} F^i_{u-v}(\delta)$$

which implies

$$\inf_{u,v\in AX} F^i_{u-v}(\varepsilon_i(\lambda)) \geq \sup_{\delta<\varepsilon_i} \inf_{u,v\in AX} F^i_{u-v}(\delta) > 1 - \lambda .$$

Let for every $i \in I$: $\varepsilon(i) = \dfrac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))}$, for every $\varepsilon > 0$.

Let $n > n_i$. Then we have

$$\frac{\varepsilon}{\prod_{s=0}^{n-1} q(f^s(i))} = \frac{\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))q(f^{n_i+1}(i))\ldots q(f^{n-1}(i))} \geq$$

$$\geq \frac{\varepsilon}{\prod_{s=0}^{n_i} q(f^s(i))Q^{n-n_i-1}(i)} = \frac{\varepsilon(i)}{Q^n(i)} .$$

The mapping t is continuous and since $t(1,1)=1$, for $\lambda \in (0,1)$ there exists $r \in (0,1)$ so that for all $x \geq r$ and all $y \geq r$ we have $t(x,y) > 1-\lambda$. Let $\varepsilon_i(r)$ be such that

$$\inf_{u,v\in AX} F^{g(i)}_{u,v}(\varepsilon_i(r)) > r .$$

Suppose that $r > 1-\lambda$. Further, let $n(i,r) \in N$ be such that

$$\frac{\varepsilon(i)}{2Q^n(i)} \geq \varepsilon_i(r) ,$$

for every $n \geq n(i,r)$.

If $n = 2k$, $p = 2m+1$ or $n = 2k+1$, $p = 2m+1$ and $n \geq n(i,r) + n_i$ it follows that:

$$F^i_{Ax_{n+p}-Ax_n}(\varepsilon) \geq F^{f^n(i)}_{Ax_o-Ax_p}\left(\frac{\varepsilon(i)}{Q^n(i)}\right)$$

where $$\varepsilon(i) = \frac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=o}^{n_i} q(f^s(i))} .$$

Furthermore

$$F^{f^n(i)}_{Ax_o-Ax_p}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \geq F^{f^n(i)}_{Ax_o-Ax_p}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right) \geq F^{g(i)}_{Ax_o-Ax_p}(\varepsilon_i(r)) > r > 1-\lambda .$$

If $n = 2k$, $p = 2m$ or $n = 2k+1$, $p = 2m$ and $n > n(i,r)+n_i$ then

$$F^i_{Ax_{n+p}-Ax_n}(\varepsilon) \geq t\left(F^{g(i)}_{Ax_1-Ax_o}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right), F^{g(i)}_{Ax_{p-1}-Ax_o}\left(\frac{\varepsilon(i)}{2Q^n(i)}\right)\right) \geq$$

$$\geq t(r,r) > 1-\lambda .$$

So we have proved that the sequence $\{Ax_n\}_{n\in N}$ is a Cauchy sequence and since X is sequentially complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} Ax_n = x^* .$$

According to the construction of the sequence $\{x_n\}_{n\in N}$ we have

$$\lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Tx_{2n} = x^* .$$

Since

$$\lim_{n\to\infty} Ax_n = x^*$$

we have the following implications

$$\lim_{n\to\infty} Ax_{2n} = x^* \Rightarrow T(\lim_{n\to\infty} Ax_{2n}) = T\overset{*}{x} \Rightarrow \lim_{n\to\infty} TAx_{2n} = Tx^* \Rightarrow$$

$$\Rightarrow A(\lim_{n\to\infty} Tx_{2n}) = Tx^* \Rightarrow A(\lim_{n\to\infty} Tx_{2n-1}) = Tx^* \Rightarrow Ax^* = Tx^*.$$

Also

$$\lim_{n\to\infty} Ax_{2n-1} = x^* \Rightarrow S(\lim_{n\to\infty} Ax_{2n-1}) = Sx^* \Rightarrow \lim_{n\to\infty} SAx_{2n-1} =$$

$$= Sx^* \Rightarrow A(\lim_{n\to\infty} Sx_{2n-1}) = Sx^* \Rightarrow A(\lim_{n\to\infty} Sx_{2n-2}) =$$

$$= Sx^* \Rightarrow Ax^* = Sx^* ,$$

and it follows that

$$Ax^* = Sx^* = Tx^* .$$

Now we shall show that Ax^* is a fixed point of the mapping A.

$$F^{i}_{Ax^*-A^2x^*}(\varepsilon) \geq F^{f(i)}_{Sx^*-TAx^*}\left(\frac{\varepsilon}{q(i)}\right) = F^{f(i)}_{Sx^*-ATx^*}\left(\frac{\varepsilon}{q(i)}\right) =$$

$$= F^{f(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon}{q(i)}\right) \geq \ldots \geq F^{f^n(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon}{\prod_{s=o}^{n-1} q(f^s(i))}\right) \geq$$

$$\geq F^{f^n(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \geq F^{g(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon(i)}{Q^n(i)}\right)$$

where $\varepsilon(i) = \dfrac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=o}^{n_i} q(f^s(i))}$, $(n > n_i)$.

We have that

$$F^{g(i)}_{Ax^*-A^2x^*}\left(\frac{\varepsilon}{Q^n(i)}\right) \to 1 \text{ , for } n \to \infty$$

and so

$$F^{i}_{Ax^*-A^2x^*}(\varepsilon) = 1$$

for all $\varepsilon > 0$ and all $i \in I$. This implies that

$$Ax^* = A^2x^* = A(Ax^*) \text{ ,}$$

i.e. Ax^* is a fixed point of the mapping A.

It is easy to see that Ax^* is a fixed point of the mappings S and T

$$S(Ax^*) = A(Sx^*) = A(Ax^*) = Ax^*$$

and

$$T(Ax^*) = A(Tx^*) = A(Ax^*) = Ax^* \text{ .}$$

Now we shall show that Ax^* is a unique fixed point of the mappings A, S and T. If we suppose that x_1 is another common fixed point of the mappings A, S and T, we have that

$$F^{i}_{Ax^*-Ax_1}(\varepsilon) \geq F^{f(i)}_{Sx^*-Tx_1}\left(\frac{\varepsilon}{q(i)}\right) = F^{f(i)}_{Ax^*-x_1}\left(\frac{\varepsilon}{q(i)}\right) \geq$$

$$\geq \ldots \geq F^{f^n(i)}_{Ax^*-Ax_1}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \geq F^{g(i)}_{Ax^*-Ax_1}\left(\frac{\varepsilon(i)}{Q^n(i)}\right) \to 1$$

for $n \to \infty$ and $\varepsilon(i) = \dfrac{Q^{n_i+1}(i)\varepsilon}{\prod_{s=o}^{n_i} q(f^s(i))}$ and for all $i \in I$, which means that

$$F^{i}_{Ax^*-x_1}(\varepsilon) = F^{i}_{Ax^*-Ax_1}(\varepsilon) = 1$$

for all $\varepsilon > 0$ and for $i \in I$. From the last equality we have that

$$Ax^* = x_1 \text{ ,}$$

which means that Ax^* is the unique fixed point for the mappings A, S and T.

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REZIME

TEOREME O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI FAMILIJE PRESLIKAVANJA U VEROVATNOSNIM LOKALNO KONVEKSNIM PROSTORIMA

$(X,\{F_i\}_{i\in I},t)$ je sekvencijalno kompletan verovatnosno lokalno konveksni prostor sa neprekidnom T-normom t. U radu je dokazana teorema u kojoj je dat potreban i dovoljan uslov za postojanje jedinstvene nepokretne tačke za dva preslikavanja S i T gde je $S: X\to X$, $T: X\to X$. Teorema glasi: Neprekidna preslikavanja S i T imaju jedinstvenu zajedničku nepokretnu tačku u X ako i samo ako postoji neprekidno preslikavanje $A: X\to SX\cap TX$ koje je komutativno sa S i T, AX je verovatnosno ograničen podskup od X i zadovoljeni su sledeći uslovi:

1. Za svako $i\in I$ postoji $q(i)>0$ i $f(i)\in I$ tako da je

$$F^{i}_{Ax-Ay}(\varepsilon)\geq F^{f(i)}_{Tx-Sy}\left(\frac{\varepsilon}{q(i)}\right)$$

za svako $\varepsilon>0$ i svako $x,y\in X$.

2. Za svako $i\in I$ postoje brojevi $n_i\in N$ i $Q(i)\in(0,1)$ tako da je

$$q(f^n(i))\leq Q(i)\leq 1 \quad \text{za svako } n>n_i$$

3. Za svako $i\in I$ postoji $g(i)\in I$ tako da je

$$F_x^{f^n(i)}(\varepsilon) > F_x^{g(i)}(\varepsilon)$$

za svako $\varepsilon > 0$, svako $x \in X$ i svako $n \in N$.

Tada postoji jedan i samo jedan elemenat $x^* \in X$ takav da je Ax^* jedinstvena nepokretna tačka za preslikavanja A, S i T.

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A NOTE ON QUOTIENT SPACES AND PARACOMPACTNESS

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ABSTRACT

The aim of the present paper is to study some properties of space $\mathcal{D}$, where $\mathcal{D}$ is an almost-upper semicontinuous decomposition of a topological space X with quotient topology.

The notation is standard except that $\alpha(A)$ will be used to denote the interior of the closure of A.

1. DEFINITIONS AND SOME KNOWN RESULTS

DEFINITION 1.1. *A subset of a space is said to be regularly open iff it is the interior of some closed set or equivalently iff it is the interior of its own closure. A set is said to be regularly closed iff it is the closure of some open set or equivalently iff it is the closure of its own interior,* |1|.

A subset is regularly open iff its complement is regularly closed.

DEFINITION 1.2. *A space X is said to be almost regular iff for any regularly closed set F and any point $x \notin F$, there exist disjoint open sets containing F and x respectively,* |10|.

DEFINITION 1.3. *A space* X *is said to be almost normal iff for every pair of disjoint sets* A *and* B *, one of which is closed and the other regularly closed, there exist disjoint open sets* U *and* V *such that* $A\subset U$, $B\subset V$, |12|.

DEFINITION 1.4 *A subset* A *of a space* X *is* α*-nearly compact (N-closed) iff every* X*-regularly open cover of* A *has a finite subcovering,* |11|.

DEFINITION 1.5. *A space* X *is locally nearly compact iff each point of* X *has an open neighbourhood* U *such that* $\bar{U}$ *is* α*-nearly compact,* |2|.

DEFINITION 1.6 *A space* X *is nearly paracompact iff every regularly open cover of* X *has a locally finite open refinement,* |12|.

DEFINITION 1.7. *Let* X *be a topological space and* A *a subset of* X*. The set* A *is* α*-nearly paracompact iff every* X*-regularly open cover of* A *has an* X*-open* X*-locally finite refinement which covers* A, |6|.

DEFINITION 1.8. *A function* $f : X \to Y$ *is said to be almost continuous iff for each point* $x \in X$ *and each open neighbourhood* V *of* f(x) *in* Y*, there exists an open neighbourhood* U *of* x *in* X *such that* $f(U)\subset \alpha(V)$, |9|.

A function is almost continuous iff the inverse image of every regularly open set is open, |9|.

DEFINITION 1.9. *A function* $f : X \to Y$ *is said to be almost closed (almost-open) iff for every regularly closed (regularly open) set* F *of* X f(F) *is closed (open) in* Y, |9|.

DEFINITION 1.10. *A decomposition* $\mathcal{D}$ *of a topological space* X *is almost-upper semicontinuous iff for each* D *in* $\mathcal{D}$ *and each regularly open set* U *containing* D *there exists an open set* V *such that* $D\subset V\subset U$ *and* V *is the union of members of* $\mathcal{D}$, |7|.

THEOREM 1.1 *A space* X *is almost regular iff for each point* $x \in X$ *and each regularly open set* V *containing* x, *there exists a regularly open set* U *such that* $x \in U \subset \bar{U} \subset V$, |10|.

THEOREM 1.2. *Let* $\mathcal{D}$ *be a decomposition of a topological space* X *and let* $\mathcal{D}$ *have a quotient topology. A decomposition* $\mathcal{D}$ *is almost-upper semicontinuous iff the projection* P *of* X *onto* $\mathcal{D}$ *is almost closed,* |7|.

THEOREM 1.3. *Let* X *be a topological space. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *whose members are* α*-nearly compact subsets of* X *and let* $\mathcal{D}$ *have a quotient topology. Then* $\mathcal{D}$ *is, respectively, Hausdorff or almost regular, provided* X *has a corresponding property,* |7|.

THEOREM 1.4. *If* A *is an* α*-nearly paracompact subset of a Hausdorff space* X *and* x *a point* $X \setminus A$, *then there are disjoint regularly open neighbourhoods of* x *and* A, |5|.

THEOREM 1.5. *If* f *is an almost closed and continuous mapping of a normal (almost normal) space* X *onto a space* Y, *then* Y *is normal (almost normal),* |4|.

THEOREM 1.6. *Let* X *be a nearly paracompact almost regular space. If* $f : X \to Y$ *is an almost continuous, almost closed surjection, such that* $f^{-1}(y)$ *is* α*-nearly compact for each point* $y \in Y$, *then* Y *is nearly paracompact almost regular,* |3|.

THEOREM 1.7. *If* f *is an almost closed and continuous mapping of a Hausdorff paracomapct space* X *onto a* T_1 *space* Y, *then* Y *is paracompact regular,* |4|.

THEOREM 1.8. *If* X *is an almost regular topological space,* A *an* α*-nearly paracompact subset,* U *a regularly open neighbourhood of* A, *then there exists a regularly open neighbourhood* V *of* A *such that* $A \subset V \subset \bar{V} \subset U$, |5|.

THEOREM 1.9. *If a mapping* $f : X \to Y$ *is almost continuous and almost closed, then for each regularly closed (regularly*

open) set F *of* Y, $f^{-1}(F)$ *is regularly closed (regularly open) in* X, |3|.

THEOREM 1.10. *A surjection mapping* $f: X \to Y$ *is almost closed iff for any subset* B *of* Y *and any regularly open set* $U \subset X$ *containing* $f^{-1}(B)$, *there exists an open set* V *in* Y *such that* $B \subset V$ *and* $f^{-1}(V) \subset U$, |8|.

2. SOME CHARACTERIZATIONS OF QUOTIENT SPACES AND PARACOMPACTNESS

THEOREM 2.1. *Let* X *be a Hausdorff space. Then for any disjoint subsets* A *and* B, *where* A *is α-nearly paracompact* B *is α-nearly compact, there exist disjoint regularly open sets* U *and* V *such that* $A \subset U$ *and* $B \subset V$.

P r o o f. For each point $x \in B$, by Theorem 1.4, there exist disjoint regularly open sets U_x and V_x such that $A \subset U_x$, $x \in V_x$. The family

$$\mathcal{V} = \{V_x : x \in B\}$$

is a cover of B by regularly open sets of X. Sinse B is α-nearly compact, there exist a finite number of points $x_1, x_2, \ldots, x_n$ in B such that

$$B \subset V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_n} .$$

Let

$$U = U_{x_1} \cap U_{x_2} \cap \ldots \cap U_{x_n} \quad \text{and} \quad V_1 = V_{x_1} \cup V_{x_2} \cup \ldots \cup V_{x_n} .$$

Now, we have $A \subsetneq U$, $B \subset V_1$ and $U \cap V_1 \neq \emptyset$. Let $V = \alpha(V_1)$. Now, U and V are regularly open subsets of X such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

THEOREM 2.2. *If* f *is an almost closed continuous mapping of a regular space* X *onto a space* Y *such that* $f^{-1}(y)$ *is α-nearly paracompact for each point* $y \in Y$, *then* Y *is regular.*

P r o o f. Since, in regular space, every α-nearly paracompact subset is α-paracompact, then $f^{-1}(y)$ is α-paracompact for each point $y \in Y$. Let $y \in Y$ and V be an open set containing y. Then $f^{-1}(V)$ is an open set in X containing $f^{-1}(y)$. Since X is regular and $f^{-1}(y)$ is α-paracompact there exists a regularly open set U in X such that

$$f^{-1}(y) \subset U \subset \bar{U} \subset f^{-1}(V).$$

Since f is almost closed, then, by Theorem 1.10, there exists an open set W in Y such that $y \in W$ and $f^{-1}(W) \subset U$. Therefore, we have

$$y \in W \subset f(U) \subset f(\bar{U}) \subset V.$$

Since f is almost closed and $\bar{U}$ is regularly closed, $f(\bar{U})$ is closed. Hence we have

$$y \in W \subset \alpha(W) \subset \bar{W} \subset V.$$

We have, that for every point $y \in Y$ and every open neighbourhood V of y there exists an open set W such that $y \in W \subset \bar{W} \subset V$, hence Y is regular.

COROLLARY 2.1. *Regularity is preserved under perfect mappings.*

THEOREM 2.3. *If f is an almost closed almost continuous mapping of an almost regular space X onto a space Y* ***such*** *that $f^{-1}(y)$ is α-nearly paracompact for each point $y \in Y$, then Y is almost regular.*

P r o o f. Let $y \in Y$ and V be a regularly open set of Y containing y. Then, since f is almost closed and almost continuous, by Theorem 1.9, $f^{-1}(V)$ is a regularly open set in X containing $f^{-1}(y)$. Since X is almost regular and $f^{-1}(y)$ is α-nearly paracompact, then, by Theorem 1.8, there exists a regularly open set U in X such that

$$f^{-1}(y) \subset U \subset \bar{U} \subset f^{-1}(V).$$

Since f is almost closed, then there exists an open set W in Y such that $y \in W$ and $f^{-1}(W) \subset U$. Therefore, we have

$$y \in W \subset f(U) \subset f(\bar{U}) \subset V.$$

Since f is almost closed, $f(\bar{U})$ is closed. Hence we have

$$y \in W \subset \alpha(W) \subset \bar{W} \subset V.$$

Since $\overline{\alpha(W)} = \bar{W}$, then $\alpha(W)$ is a regularly open set containing y such that $y \in \alpha(W) \subset \overline{\alpha(W)} \subset V$, hence, by Theorem 1.1, Y is almost regular.

COROLLARY 2.2. *If f is an almost closed, almost continuous mapping of an almost regular spaces X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each point $y \in Y$, then Y is almost regular,* |5|.

P r o o f. Every α-nearly compact is α-nearly paracompact.

THEOREM 2.4. *Let X be a topological space. Let $\mathcal{D}$ be an almost-upper semicontinuous decomposition of X whose members are α-nearly paracompact subsets of X and let $\mathcal{D}$ have a quotient topology. If X is regular, then the projection P of a space X onto a space $\mathcal{D}$ is closed. If X is Hausdorff (regular, almost regular), then $\mathcal{D}$ is T_1 (regular, almost regular).*

P r o o f. Let X be a regular. Now, we have that for every point $A \in \mathcal{D}$, A is an α-nearly paracompact subset in X. Since every α-nearly paracompact subset in regular space is α-paracompact, then A is an α-paracompact subset in X. Let U be an open neighbourhood of A in X. Since A is α-paracompact, there exists an open neighbourhood W of A such that

$$A \subset W \subset \bar{W} \subset U.$$

Now, $\alpha(W)$ is a regularly open neighbourhood of A. Since $\mathcal{D}$ is almost-upper semicontinuous, then there exists an open set V such that $A \subset V \subset \alpha(W)$ and V is the union of members of $\mathcal{D}$. Now, we have that for each $A \in \mathcal{D}$ and each open set U containing A, there exists an open set V such that $A \subset V \subset U$ and V is the union of members of $\mathcal{D}$, hence $\mathcal{D}$ is an upper semicontinuous decomposition of X. Since $\mathcal{D}$ is an upper semicontinuous decomposition of

X, the projection P of a space X onto a space $\mathcal{D}$ is closed. If X is regular (almost regular) then $\mathcal{D}$ is, by Theorem 2.2 (Theorem 2.3) regular (almost regular).

Now, let X be a Hausdorff space. Let A be any point of $\mathcal{D}$. Then the set A is an α-nearly paracompact subset in X. By Theorem 1.4 A is closed in X. Since a subset F of $\mathcal{D}$ is closed iff $P^{-1}(F)$ is closed in X, then the point $A \in \mathcal{D}$ is closed in $\mathcal{D}$, hence $\mathcal{D}$ is T_1.

THEOREM 2.5. *Let X be a topological spaces. Let $\mathcal{D}$ be an almost-upper semicontinuous decomposition of X whose members are α-nearly paracompact subset of X and let $\mathcal{D}$ have a quotient topology. If X is a Hausdorff paracompact space, then $\mathcal{D}$ is paracompact regular.*

P r o o f. Since X is a Hausdorff paracompact space, then X is regular. By Theorem 2.4, $\mathcal{D}$ is regular. Since X is a Hausdorff space, then $\mathcal{D}$ is T_1. Since every Hausdorff paracompact space is normal, X is normal. Since the projection of a space X onto a space $\mathcal{D}$ is almost closed and continuous, it follows easily from Theorem 1.7, that $\mathcal{D}$ is a regular and paracompact space.

THEOREM 2.6. *Let X be a topological spaces. Let $\mathcal{D}$ be an almost-upper semicontinuous decomposition of X whose members are α-nearly compact subsets of X and let $\mathcal{D}$ have a quotient topology. If X is paracompact regular, then $\mathcal{D}$ is paracompact regular.*

P r o o f. Since the members of $\mathcal{D}$ are α-nearly compact subsets of X and the projection P of X onto $\mathcal{D}$ is almost closed and continuous, then $\mathcal{D}$ is paracompact regular.

THEOREM 2.7. *Let X be a topological space. Let $\mathcal{D}$ be an almost-upper semicontinuous decomposition of X and let $\mathcal{D}$ have a quotient topology. If X is normal (almost normal), then $\mathcal{D}$ is normal (almost normal).*

P r o o f. Since the projection P of a space X onto a space $\mathcal{D}$ is almost closed and continuous, then, by Theorem 1.5 $\mathcal{D}$ is normal (almost normal).

THEOREM 2.8. *Let* X *be a topological space. Let* $\mathcal{D}$ *be an almost-upper semicontinuous decomposition of* X *whose members are* α*-nearly compact subsets of* X *and let* $\mathcal{D}$ *have a quotient topology. If* X *is nearly paracompact almost regular, then* $\mathcal{D}$ *is nearly paracompact almost regular.*

P r o o f. Since the members of $\mathcal{D}$ are α-nearly compact subsets of X and the projection P of X onto $\mathcal{D}$ is almost closed and continuous, then, by Theorem 1.6, $\mathcal{D}$ is nearly paracompact almost regular.

LEMMA 2.1. *If* $f : X \to Y$ *is an almost closed almost continuous mapping of a space* X *onto a space* Y, *then the image of every* α*-nearly compact subset in* X *is an* α*-nearly compact subset in* Y.

P r o o f. Let K be any α-nearly compact subset in X. Let

$$\mathcal{U} = \{U_\alpha : \alpha \in I\}$$

be any Y-regularly open cover of f(K). Then

$$\{f^{-1}(U_\alpha) : \alpha \in I\}$$

is an X-regularly open cover of K. Since K is α-nearly compact, there exists a finite subset I_o of I such that

$$K \subset \bigcup\{f^{-1}(U_\alpha) : \alpha \in I_o\}.$$

Now, we have

$$f(K) \subset \bigcup\{U_\alpha : \alpha \in I_o\},$$

hence f(K) is α-nearly compact.

THEOREM 2.9. *Let* $f : X \to Y$ *be an almost closed almost continuous mapping of a space* X *onto a space* Y *such that* $f^{-1}(y)$ *is* α*-nearly compact for each point* $y \in Y$. *If* X *is a Hausdorff*

locally nearly compact space, then Y *is locally nearly compact Hausdorff.*

P r o o f. Y is a Hausdorff space. We shall show that Y is locally nearly compact. Let y be any point of Y. Since X is locally nearly compact Hausdorff, for each point $x \in f^{-1}(y)$, there exists a regularly open neighbourhood K_x such that $\bar{K}_x$ is α-nearly compact. Now, the family

$$\mathcal{K} = \{K_x : x \in f^{-1}(y)\}$$

is an X-open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is α-nearly compact, then there exist a finite number of points $x_1, x_2, \ldots, x_n$ in $f^{-1}(y)$ such that

$$f^{-1}(y) \subset \cup\{K_{x_i} : i=1,2,\ldots,n\}.$$

Let

$$K = \cup\{\bar{K}_{x_i} : i=1,2,\ldots,n\}.$$

$f^{-1}(y) \subset K^o$. Since f is almost closed, then there exists an open set V_y containing y such that $f^{-1}(V_y) \subset K^o$. Hence, we have

$$y \in V_y \subset f(K^o) \subset f(K).$$

Since f is almost continuous and almost closed and K is α-nearly compact, then f(K) is α-nearly compact. Since Y is Hausdorff f(K) is closed. Therefore we have

$$\bar{V}_y \subset f(K).$$

Since, every Y-regularly closed subset of an α-nearly compact subset is α-nearly compact, then $\bar{V}_y$ is α-nearly compact.

Now, V_y is a Y-open neighbourhood of y such that $\bar{V}_y$ is α-nearly compact, hence Y is locally nearly compact.

COROLLARY 2.3. *Let* $f: X \to Y$ *be an almost closed almost continuous surjection with N-closed point inverses. If* X *is locally compact Hausdorff, then* Y *is locally nearly compact Hausdorff,* |*8*|.

THEOREM 2.10. *Let X be a topological space. Let $\mathcal{D}$ be an almost-upper semicontinuous decomposition of X whose members are α-nearly compact subsets of X and let $\mathcal{D}$ have a quotient topology. If X is locally nearly compact Hausdorff, then $\mathcal{D}$ is locally nearly compact Hausdorff.*

P r o o f. Since the members of $\mathcal{D}$ are α-nearly compact subsets of X and the projection P of X onto $\mathcal{D}$ is almost closed and continuous, then, by Theorem 2.9, $\mathcal{D}$ is locally nearly compact Hausdorff.

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REZIME

KOLIČNIK PROSTORI I PARAKOMPAKTNOST

Cilj rada je ispitivanje osobina prostora $\mathcal{D}$, gde je $\mathcal{D}$ skoro polu-neprekidno razlaganje prostora X sa količnik topologijom.

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ON THE PREDICTION OF A FUNCTIONAL OF GAUSSIAN RANDOM PROCESS

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Let $\{\xi(t),\ t\in T\}$ be a real Gaussian process (centered at the expectation: $E\xi(t)=0$) and let η be a integrable functional measurable with respect to σ-field $F(T)$ generated by $\{\xi(t),\ t\in T\}$. It is well-known that the conditional expectation $\hat{\xi}(t)=E(\xi(t)|F(S)|$, $S\subset T$, coincides with the projection of $\xi(t)$ on the Hilbert space $H^1(S)$ spanned on $\{\xi(t),\ t\in S\}$.

THEOREM. *The functional* $\hat{\eta}=E(\eta|F(S))$ *is measurable with respect to* σ*-field* $F_{\hat{\xi}}$ *generated by* $\{\hat{\xi}(t),\ t\in T\}$.

P r o o f. The conditional distribution (given $F(S)$) of η is determined by the family of the conditional distributions of the vectors $(\xi(t_1),\dots,\xi(t_n))$, $t_1,\dots,t_n\in T$. The conditional distribution of $(\xi(t_1),\dots,\xi(t_n))$ is Gaussian, so it is determined by the mean vector $(\xi(t_1),\dots,\xi(t_n))$ and the covariance matrix

$$B=\|b(t_i,t_j)\|,\quad i,j=1,\dots,n,$$

$$b(t_i,t_j)=E((\xi(t_i)-\hat{\xi}(t_i))(\xi(t_j)-\hat{\xi}(t_j))|F(S)).$$

But $\xi(t)-\hat{\xi}(t)$ is independent (for Gaussian process) of $F(S)$. So $b(t_i,t_j)$ is the constant $E(\xi(t_i)-\hat{\xi}(t_i))(\xi(t_j)-\hat{\xi}(t_j))$. In this way the conditional distribution of η depends only of $\hat{\xi}(t)$, $t\in T$, and its conditional expectation $\hat{\eta}$ depends only of $\hat{\xi}(t)$, $t\in T$.

Theorem is closely related to |2|, pp.73-78, but puts in evidence the evaluation of $\hat{\eta}$ by $\{\hat{\xi}(t)\}$.

Example 1. Let $H^n(S)$ be the linear closure of all polynomials of the variables $\xi(t)$, $t \in S$, the degree not greated than n. It is shown in |2| that $\eta \in H^n(T)$ implys $\hat{\eta} \in H^n(S)$. We precise this result showing that $\hat{\eta}$ belongs to the linear closure of all polynomias of the variables $\hat{\xi}(t)$, $t \in T$, the degree not greated than n. For this it is enough to see that

$E(\xi(t_1)\ldots\xi(t_n)\,|\,F(S))$ is a polynomials of $\hat{\xi}(t_1),\ldots \hat{\xi}(t_n)$ of the degree n:

$$E(\xi(t_1)\ \ldots\ \xi(t_n)\,|\,F(S)) =$$

$$= \frac{1}{(2\pi)^{n/2}(\det B)^{1/2}} \int\ldots\int x_1\ldots x_n \exp\{-\frac{1}{2}\sum_{i,j} B_{ij}(x_i - \hat{\xi}(t_i))(x_j-\hat{\xi}(t_j)))\}dx_1\ldots dx_n = \frac{1}{(2\pi)^{n/2}(\det B)^{1/2}} \int \ldots$$

$$\ldots\int \prod_{k=1}^{n} (u_k+\hat{\xi}(t_k))\exp\{-\frac{1}{2}\sum_{i,j} B_{ij}u_iu_j\}du_1\ldots du_n =$$

$$= P_n(\hat{\xi}(t_1),\ldots \hat{\xi}(t_n))\ .\ (B_{ij} \text{ is the cofactor of } b(t_i,t_j))\ .$$

Observe that

$$\frac{\partial}{\partial x_k} P_n(x_1,\ldots,x_k,\ldots,x_n) = P_{n-1}(x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_n)\,.$$

APPLICATION TO THE PREDICTION

Let $T = [t_o,\infty)$ $S = [t_o,s]$, $t_o < s$. Then for fixed $t, t > s$, $\hat{\xi}(s,t) = E(\xi(t)\,|\,F_s)$, (we put $F(S) = F_s$), is the best (in the sence of the minimal variance) prediction of $\xi(t)$ by $\{\xi(u),\ u \in [t_o,s]\}$. $\hat{\xi}(s,t)$ coincides, for Gaussian process, with the best linear prediction which is widely elaborated for stationary process $\{\xi(t), -\infty < t < \infty\}$. In the terms of predication problem Theorem says

that the prediction $\hat{\eta}$ of an integrable functional η of $\{\xi(u), u \in [s,\infty)\}$ by $\{\xi(u), u \in [t_o, s]\}$ is the functional of $\{\hat{\xi}(s,u), u \in [s,\infty]\}$

<u>Example 2.</u> Prediction of the time over a level by Gaussian process. Supposing the continuity of $\{\xi(t), t \geq 0\}$ the functional $\eta = \int_s^t I(\xi(u) > c)du$, ($I(.)$ is the indicator function), is the time over the level c by the process $\{\xi(u)\}$ during the time $[s,t]$. The prediction of η by $\{\xi(u), u \leq s\}$ is

$$\hat{\eta} = E(\eta | F_s) = \int_s^t P(\xi(u) > c | F_s)du = \int_s^t \left[1-\Phi\left(\frac{c-\hat{\xi}(s,u)}{\sqrt{b(u,u)}}\right)\right]du,$$

where $\Phi(.)$ is the distribution function of a standard Gaussian variable.

<u>Example 3.</u> The problem of the prediction of the process $\{\zeta(t), t \geq o\}$, $\zeta(t) = f(\xi(t))$, where $f(.)$ is a non-random function, is posed in |1|. In the case $f(x) = x^2$ and the differentiable process $\{\xi(t), t \geq 0\}$ for which

$$\hat{\xi}(s,t) = \sum_{j=o}^{N-1} a_j(s,t)\xi^{(j)}(s) ,$$

(such process belongs to so called N-tiple Markov processes), the explicit formula for $\hat{\zeta}(s,t)$ in the terms of $\zeta(s),\ldots \zeta^{(N-1)}(s)$ is given.

But generaly, because the conditional distribution of $\xi(t)$ given F_s is Gaussian with the parameters $\hat{\xi}(s,t)$ and $b(t,t)$, we have simple

$$\hat{\zeta}(s,t) = E(f(\xi(t))|F_s) = \frac{1}{\sqrt{2\pi b(t,t)}} \int f(x)\exp\left\{-\frac{1}{2}\frac{(x-\hat{\xi}(t,s))^2}{b(t,t)}\right\}dx =$$

$$= g(\hat{\xi}(t,s)) .$$

For instance, if $\{\xi(t)\}$ is as in |1| and $f^{-1}(.)$ is the differentiable function, we find $\hat{\zeta}(s,t)$ in terms of $\zeta(s),\ldots\zeta^{(N-1)}(s)$.

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REZIME

O PREDVIDJANJU FUNKCIONALA GAUSOVOG SLUČAJNOG PROCESA

Pokazuje se da je predvidjanje funkcionala Gausovog procesa $\{\xi(t),\ t \in T\}$ funkcional od linearnog predvidjanja $\{\hat{\xi}(s,t),\ t \in T\}$.

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A UNIFORMLY CONVERGENT SCHEME WITH QUASI-CONSTANT FITTING FACTORS*

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1. INTRODUCTION

Consider the problem

$$\text{(BVP)}\qquad \begin{aligned} Lu(x) := -\varepsilon^2 u''(x) + g^2(x)u(x) &= f(x), \quad x \in (0,1) \\ u(o) = A, \quad u(1) &= B. \end{aligned}$$

Here ε is a small positive constant, A and B are given constats, g and f are in $C^1[0,1]$, $g(o) = g(1)$ and $g(x) \geq \gamma$ on $[0,1]$ for some positive constant γ.

Under these assumptions (BVP) has a unique solution $u \in C^3[0,1]$ which in general displays a boundary layer at $x = o$ and $x = 1$ for "small" ε, |1|,|2|,|3|.

We want to solve the above problem by difference approximations on a non-uniform mesh which has more mesh points in the boundary regions than away from the boundary layers. In the case when the mesh is uniform our difference scheme is the same as in |2|.

Throughout the paper we shall let $C, C_1, \ldots$ denote positive constants that may take different values in different formulas, but that are always independent of ε and hk_i, $i=1,2,\ldots,n$. We assume that the parameter ε satisfies $o < \varepsilon \leq \varepsilon_o$ where ε_o is a positive constant.

2. DIFFERENCE APPROXIMATION FOR (BVP).

Consider a non uniform mesh

$$I_h = \{x_o = o,\ x_j = x_{j-1} + hk_k :\ j=1,2,\ldots,n\}$$

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with $n=2m$, $m\in N$, $k_j\in R$, $j=1,2,\dots,n$, $h^{-1}=\sum_{j=1}^{n} k_j$.

We want to choose $k_1,k_2,\dots,k_n$ in such a way that the following conditions are satisfied:

$$1\le k_j\le k_{j+1}\,,\qquad j=1,2,\dots,m-1$$
$$k_m=k_{m+1}$$
$$1\le k_{j+1}\le k_j\,,\qquad j=m+1,\,m+2,\dots,n-1\,.$$

Let $\delta=g(o)/\varepsilon$ and for $i\in\{1,2,\dots,n-1\}$

$$(1)\qquad \alpha_i=\begin{cases}-k_i & \text{if } i\le m\\ k_{i+1} & \text{if } i>m\end{cases}\,,\quad \beta_i=\begin{cases}k_{i+1} & \text{if } i\le m\\ -k_i & \text{if } i>m\end{cases}.$$

Let u_i denote the approximate value (to be determined) for $u(x_i)$, $i=0,1,\dots,n$. We approximate (BVP) by

$$u_o=A\,,$$
$$\text{(DBVP)}\qquad L^h u_i:=L_o^h u_i+g^2(x_i)u_i=f(x_i),\ i=1,2,\dots,n-1$$
$$u_n=B\,,$$

where

$$L_o^h u_i:=(da_i+(1-d)c_i)u_{i-1}+b_iu_i+$$
$$+((1-d)a_i+dc_i)u_{i+1}\,,$$

$$d=\begin{cases}1 & \text{if } i\le m\\ 0 & \text{if } i>m\end{cases}\,,$$

$$(2)\qquad a_i=-\frac{g^2(o)}{S}\sinh\delta\beta_i h,\quad c_i=\frac{g^2(o)}{S}\sinh\delta\alpha_i h\,,$$
$$b_i=-a_i-c_i\,,$$

$$(3)\qquad S=\sinh\delta(\beta_i-\alpha_i)h-\sinh\delta\beta_i h+\sinh\delta\alpha_i h\,.$$

In the Appendix it is proved that $a_i<0$, $c_i<0$, $b_i>0$.

The above difference approximation for (BVP) is derived as follows. For a fixed $i\in\{1,2,\dots,n-1\}$ consider a linear system

(4) $L_o^h F_j(x_i) = -\varepsilon^2 F_j''(x_i)$, $j=1,2,3$

where $F_1(x)=1$, $F_2(x)=\exp(\delta x)$, $F_3(x)=\exp(-\delta x)$. The solution of this system in given by (2) and (3).

THEOREM. *Suppose* $g, f \in C^1[0,1]$, $g(o)=g(1)$ *and* $g(x) \geq \gamma > o$ *on* $[0,1]$. *Then the solution* u *of (BVP) and the solution* u_i *of (DBVP) satisfy*

(5) $|u(x_i) - u_i| \leq Chk_m$, $i=0,1,\dots,n$

where C *is indepedent of* i, h, k_m *and* ε.

P r o o f: We use in part: the notation, technique and some results from |2|. Since $g(o)=g(1)$ the solution of (BVP) can be written as

$$u(x) = v(x) + w(x) + z(x)$$

where

$$u_o(x) = f(x)/g^2(x), \quad v(x) = p\exp(-\delta x), \quad w(x) = g\exp(\delta x),$$

$$p = (A-u_o(o) + (u_o(1) - B)\exp(-\delta))(1-\exp(-2\delta))^{-1}$$

$$q = (B-u_o(1) + (u_o(o) - A)\exp(-\delta))(1-\exp(-2\delta))^{-1}\exp(-\delta),$$

and $z(x)$ is the solution of

$$Lz(x) = f(x) - Lv(x) - Lw(x), \quad x \in (0,1)$$

$$z(o) = u_o(o), \quad z(1) = u_o(1).$$

For $z(x)$ we obtain, |2|,

$$|z^{(j)}(x)| \leq C(1+\varepsilon^{2-j}), \quad j=0,1,2.$$

Defining the mesh functions $\{v_i\}$, $\{w_i\}$, $\{z_i\}$ by

$$L^h v_i = Lv(x_i), \quad v_o = v(o), \quad v_n = v(1),$$

$$L^h w_i = Lw(x_i), \quad w_o = w(o), \quad w_n = w(1),$$

$$L^h z_i = Lz(x_i), \quad z_o = z(o). \quad z_n = z(1),$$

we have

$$u_i = v_i + w_i + z_i.$$

Using (4) and (DBVP) it is easy to show that $v_i = v(x_i)$ and $w_i = w(x_i)$ for all $o \leq i \leq n$. Now, using these facts, we obtain

$$(6)\quad L^h(u(x_i)-u_i) = L^h(z(x_i)-z_i) = L^h z(x_i) - Lz(x_i) \; .$$

We can write the (DBVP) as

$$A_h u_h = f_h \; ,$$

where

$$A_h = \begin{bmatrix} 1 & & & & & & \\ a_1 & b_1+g^2(x_1) & c_1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & a_m & b_m+g^2(x_m) & c_m & & \\ & & & c_{m+1} & b_{m+1}+g^2(x_{m+1}) & a_{m+1} & \\ & & & & \ddots & \ddots & \ddots \\ & & & & c_{n-1} & b_{n-1}+g^2(x_{n-1}) & a_{n-1} \\ & & & & & & 1 \end{bmatrix},$$

$$u_h = [u_o,u_1,\dots,u_{n-1},u_n]^T, \qquad f_h = [A,f(x_1),\dots,f(x_{n-1}),B]^T \; .$$

If we denote $A_h = [(A_h)_{ij}] \in R^{n+1,n+1}$ we have

$$|(A_h)_{ii}| = \begin{cases} 1 & \text{for } i=1 \text{ and } i=n+1 \\ b_{i-1}+g^2(x_{i-1}) & \text{for } i=2,3,\dots,n \end{cases}$$

$$\sum_{\substack{j=1\\ j\neq i}}^{n+1} |(A_h)_{ij}| = \begin{cases} 0 & \text{for } i=1 \text{ and } i=n+1 \\ -a_{i-1}-c_{i-1} & \text{for } i=2,3,\dots,n \end{cases} .$$

Since

$$\min_i(|(A_h)_{ii}| - \sum_{\substack{j=1\\ j\neq i}}^{n+1} |(A_h)_{ij}|) = \min_i(b_i+a_i+c_i+g^2(x_i),1) =$$

$$= \min_i(g^2(x_i),1) \geq \min(\gamma^2,1) > 0 \; ,$$

we can now apply Theorem 1. from [4] and we conclude that A_h is non singular and

$$(7)\qquad \|A_h^{-1}\|_\infty \leq \frac{1}{\min(\gamma^2,1)} \; .$$

Now from (6) we obtain

(8) $$\|y\|_\infty \le \|A_h^{-1}\|_\infty \|r\|_\infty ,$$

where $y = [y_o, y_1, \ldots, y_n]^T$, $r = [r_o, r_1, \ldots, r_n]^T$ and

$$y_i = u(x_i) - u_i \ , \quad r_i = L^h z(x_i) - Lz(x_i), \ i=0,1,\ldots,n.$$

We want to prove that

(9) $$\|r\|_\infty \le C\, h\, k_m$$

and then (5) follows directly from (7) and (8).

From (BVP) and (DBVP) follows

$$r_i = \varepsilon^2 (z''(x_i) - \frac{\delta^2}{S} \sinh\delta\beta_i h (z(x_{i-1}) + (\rho + \frac{\alpha_i - \beta_i}{\beta_i}) z(x_i)$$

$$- (\rho + \frac{\alpha_i}{\beta_i}) z(x_{i+1})))$$

where

$$= \frac{\sinh\delta\alpha_i h}{\sinh\delta\beta_i h} - \frac{\alpha_i}{\beta_i} .$$

Since

$$z(x_{i-1}) + (\rho + \frac{\alpha_i - \beta_i}{\beta_i}) z(x_i) - (\rho + \frac{\alpha_i}{\beta_i}) z(x_{i+1}) =$$

$$= \frac{\alpha_i(\alpha_i - \beta_i)}{2} \left(\frac{2z(x_{i-1})}{\alpha_i(\alpha_i - \beta_i)} + \frac{2z(x_i)}{\alpha_i\beta_i} + \frac{2z(x_{i+1})}{\beta_i(\beta_i - \alpha_i)} \right) +$$

$$+ \rho\, (z(x_i) - z(x_{i+1})) ,$$

and (see the Appendix)

(10) $$\frac{2z(x_{i-1})}{\alpha_i(\alpha_i - \beta_i)} + \frac{2}{\alpha_i\beta_i} z(x_i) + \frac{2z(x_{i+1})}{\beta_i(\beta_i - \alpha_i)} = h^2 z''(\tau_i) ,$$

$$\tau_i \in (x_{i-1}, x_{i+1}) ,$$

$$z''(x_i) = z''(\tau_i) + (x_i - \tau_i) z'''(\eta_i) , \quad \eta_i \in (\min(x_i, \tau_i), \max(x_i, \tau_i)),$$

$$z(x_i) - z(x_{i+1}) = -z'(\theta_i)(x_{i+1} - x_i) = -z'(\theta_i)\, h k_{i+1} ,$$

$$\theta_i \in (x_i, x_{i+1})$$

we have

$$r_i = \varepsilon^2(z''(x_i) - \frac{\delta^2}{S}\sinh\delta\beta_i h(\frac{\alpha_i(\alpha_i-\beta_i)}{2}h^2 z''(\tau_i) - $$
$$- \rho z''(\theta_i)hk_{i+1})) ,$$

and $r_i = \varepsilon^2(z''(\tau_i)(1-P)+(x_i-\tau_i)z'''(\eta_i)+\rho Phk_{i+1}z'(\theta_i))$,

where

$$\text{(11)} \qquad P = \frac{\alpha_i(\alpha_i-\beta_i)\delta^2h^2}{2S}\sinh\delta\beta_i h .$$

Now we see that $(\tau_i \in (x_{i-1},x_{i+1}), |x_i-\tau_i| \leq hk_{i+1})$

$$|r_i| \leq \varepsilon^2 \sup_{x\in[0,1]} |z''(x)||1-P| + \varepsilon^2 \sup_{x\in[0,1]} |z'''(x)|hk_{i+1}$$
$$+ \varepsilon^2 \sup_{x\in[0,1]} |z'(x)|\frac{\delta^2}{\beta_i S}(\beta_i\sinh\delta\alpha_i h - \alpha_i\sinh\delta\beta_i h)kh_{i+1} ,$$

and since (see the Appendix) $0 \leq \beta_i\sinh\delta\alpha_i h - \alpha_i\sinh\delta\beta_i h \leq \beta_i S$, $\frac{(\delta\beta_i h)^2}{4\sinh^2\frac{\delta\beta_i h}{2}} \leq P \leq 1 + \frac{(\delta\beta_i h)^2}{4}$, we obtain $(\max(k_i: i=1,2,\dots,n) = k_m)$

$$|r_i| \leq \varepsilon^2 C_1|1-P| + \varepsilon^2 C_2(1+1/\varepsilon)hk_{i+1} + \varepsilon^2\delta^2 C_3(1+\varepsilon)hk_{i+1} ;$$
$$|r_i| \leq 0.25\varepsilon^2\delta^2 C_1(hk_{i+1})^2 + C_4hk_{i+1} + C_5hk_{i+1} \leq Chk_m ,$$

i.e. (9) is proved.

REMARK 1. In the case $k_i=1$, $i=1,2,\dots,n$ we have a uniform mesh and our scheme is the same as in [2]. In this case the estimate $|u(x_i)-u_i| \leq Ch^2$, $i=0,1,\dots,n$ holds with the assumption that $g,f \in C^2[0,1]$, and then the proof of the theorem is not as complicated as in our case.

REMARK 2. From (5) it follows that the solution of (DBVP) converges to that of (BVP) as $\max_i(x_i-x_{i-1}) \to 0$, as $n \to \infty$ uniformly in ε.

3. APPENDIX

In this section we denote for a fixed $i \in \{1,2,\dots,n-1\}$ α_i by α and β_i by β.

LEMMA 1. *For a_i, b_i, c_i from (2), we have*

$$a_i < 0,\ c_i < 0,\ b_i > 0,\ i=1,2,\dots,n-1 .$$

P r o o f: We shall consider three cases: 1) $i = m$; 2) $i < m$; 3) $i > m$.

1) In this case $-\alpha = \beta = hk_m$ and

$$a_i = c_i = -0.5\, b_i = -(0.5\delta)^2 \sinh^{-2}(0.5\delta\beta h) < 0 .$$

2) In this case we have $1 \le -\alpha \le \beta$ and

$$S = S(\alpha,\beta) = \sinh\delta(\beta-\alpha) - \sinh\delta\beta h + \sinh\delta\alpha h$$

$$= \sum_{k=1}^{\infty} \frac{(\delta h)^{2k-1}}{(2k-1)!} \left((\beta-\alpha)^{2k-1} - \beta^{2k-1} + \alpha^{2k-1}\right)$$

$$= \sum_{k=2}^{\infty} \frac{(\delta h)^{2k-1}}{(2k-1)!} \sum_{j=1}^{2k-2} \binom{2k-1}{j} \beta^{2k-1-j}(-\alpha)^j > 0 ,$$

since $\delta h > 0$. Now we have

$$a_i = -\frac{g^2(o)}{S} \sinh\delta\beta h < 0, \quad (\beta > 0)$$

$$c_i = \frac{g^2(o)}{S} \sinh\delta\alpha h < 0, \quad (\alpha < 0)$$

$$b_i = -(a_i + c_i) > 0 .$$

3) In this case $1 \le -(-\alpha) \le -\beta$ and $S(\alpha,\beta) = -S(-\alpha,-\beta)$. Since $S(-\alpha,-\beta) > 0$, this is proved in 2), we obtain $S(\alpha,\beta) < 0$ and $a_i < 0$ $(\beta < 0)$, $c_i < 0$ $(\alpha > 0)$.

LEMMA 2. *If $z \in C^2[0,1]$ then (10) is satisfied.*

P r o o f: We have for $i \le m$

$$z(x_{i-1}) = z(x_i+\alpha h) = z(x_i) + \alpha h z'(x_i) + 0.5(\alpha h)^2 z''(\tau_1) ,$$
$$z(x_{i+1}) = z(x_i+\beta h) = z(x_i) + \beta h z'(x_i) + 0.5(\beta h)^2 z''(\tau_2) ,$$
$$\tau_1 \in (x_{i-1},x_i) , \quad \tau_2 \in (x_i,x_{i+1}) ,$$

and

$$\frac{2z(x_{i-1})}{\alpha(\alpha-\beta)} - \left(\frac{2}{\alpha(\alpha-\beta)} + \frac{2}{\beta(\beta-\alpha)}\right) z(x_i) + \frac{2z(x_{i+1})}{\beta(\beta-\alpha)} =$$

$$h^2\left(\frac{-\alpha}{\beta-\alpha} z''(\tau_1) + \frac{\beta}{\beta-\alpha} z''(\tau_2)\right) = h^2 z''(\tau) , \quad \tau \in (x_{i-1},x_{i+1}).$$

For $i > m$ the proof is the same.

LEMMA 3. *For* S *from (2) and* P *from (11) and for all* $1 \le i \le n-1$

(i) $0 \le \beta\sinh\delta\alpha h - \alpha\sinh\delta\beta h \le \beta S$,

(ii) $\dfrac{(\delta\beta h)^2}{4\sinh^2\frac{\delta\beta h}{2}} \le P \le 1 + \dfrac{(\delta\beta h)^2}{4}$,

(iii) $|1-P| \le \left(\dfrac{\delta\beta h}{2}\right)^2$.

P r o o f: First shall we consider the case $i \le m$.

(i): From $\beta \ge -\alpha \ge 1$ follows

$$-\alpha\beta(\beta^{2k-2} - \alpha^{2k-2}) \ge 0 , \quad \text{for } k \ge 1 ,$$

$$\sum_{k=1}^{\infty} \frac{(\delta h)^{2k-1}}{(2k-1)!} (\beta\alpha^{2k-1} - \alpha\beta^{2k-1}) \ge 0$$

and $\beta\sinh\delta\alpha h - \alpha\sinh\delta\beta h \ge 0$.

Now we have $(S\beta > 0)$ $\beta\sinh\delta\alpha h - \alpha\sinh\delta\beta h - \beta S =$

$$\sum_{k=2}^{\infty} \frac{(\delta h)^{2k-1}}{(2k-1)!} \left(-\alpha\beta(\beta^{2k-2} - \alpha^{2k-2}) - \beta \sum_{j=1}^{2k-2} \binom{2k-1}{j} \beta^{2k-1-j}(-\alpha)^j\right).$$

Since $-\alpha\beta(\beta^{2k-2}-\alpha^{2k-2})-\beta\beta^{2k-2}(-\alpha)(2k-1)-\beta\beta(-\alpha)^{2k-2}(2k-1) -$
$\beta \sum_{j=2}^{2k-3} \binom{2k-1}{j} \beta^{2k-1-j}(-\alpha)^j \le 0$, for $k \ge 2$, (ii) follows directly.

(ii) Using the elementary properties of hyperbolic functions we obtain

$$P = P(\alpha,\beta) = 0.5\alpha(\alpha-\beta)\delta^2h^2 \frac{\cosh\frac{\delta\beta h}{2}}{\cosh\frac{\delta(\beta-2\alpha)h}{2} - \cosh\frac{\delta\beta h}{2}} .$$

Now we have

$$\frac{\partial P}{\partial\alpha}(\alpha,\beta) = 0.5\delta^2h^2\cosh\frac{\delta\beta h}{2}\; Q/(\cosh\frac{\delta(\beta-2\alpha)h}{2} - \cosh\frac{\delta\beta h}{2})^2 ,$$

where

$$Q = -(\beta-2\alpha)(\cosh\frac{\delta(\beta-2\alpha)h}{2} - \cosh\frac{\delta\beta h}{2}) + \alpha(\alpha-\beta)\delta h\sinh\frac{\delta(\beta-2\alpha)h}{2} ,$$

and we see that $\frac{\partial P}{\partial\alpha}(\alpha,\beta) > 0$ if and only if $Q > 0$, for $\beta \geq -\alpha \geq 1$.

We write Q in the form

$$Q = -(\beta-2\alpha)\sum_{k=o}^{\infty}\frac{(0.5\delta h)^{2k}}{2k!}((\beta-2\alpha)^{2k} - \beta^{2k}) +$$

$$+ \alpha(\alpha-\beta)\sum_{k=1}^{\infty}\frac{(0.5\delta h)^{2k-1}}{(2k-1)!}(\beta-2\alpha)^{2k-1} ,$$

$$Q = -(\beta-2\alpha)\sum_{k=1}^{\infty}\frac{(0.5\delta h)^2}{(2k)!}R_k .$$

where $R_k = (\beta-2\alpha)^{2k}(1 - \frac{4k\alpha(\alpha-\beta)}{(\beta-2\alpha)^2}) - \beta^{2k}$.

We immediately see that $R_1 = 0$, $R_2 = -16\alpha^2(\alpha-\beta)^2 < 0$, and that $R_{k+1} = \beta^2R_k - 16\alpha^2k(\beta-2\alpha)^{2k-2}(\alpha-\beta)^2$, $k=1,2,\ldots$. Now, from $R_k < 0$ follows $R_{k+1} < 0$ and we conclude that $\frac{\partial P}{\partial\alpha}(\alpha,\beta) > 0$ for $\beta \geq -\alpha \geq 1$. From this we obtain $(\alpha \geq -\beta)$

$$P(\alpha,\beta) \geq P(-\beta,\beta) = \frac{(\delta\beta h)^2}{2}\sinh^{-2}(\frac{\delta\beta h}{2}),$$

which is the first part of (ii).

Now we want to prove $P(\alpha,\beta) \leq 1 + (0.5\delta\beta h)^2$. This inequality is equivalent to $T \leq 0$, where

$$T = 0.5\alpha(\alpha-\beta)\cosh(0.5\delta\beta h) - (1+(0.5\delta\beta h)^2(\cosh(0.5\delta(\beta-2\alpha)h) - \cosh(0.5\delta\beta h)) .$$

Using Taylor expansions we obtain

$$T = 0.5\alpha(\alpha-\beta)\delta^2h^2 + \alpha(\alpha-\beta)\delta^2h^2(-8-\delta^2\beta^2h^2)/16 + \sum_{k=2}^{\infty} \frac{(0.5\delta h)^2 L_k}{(2k)!},$$

where for $k \geq 2$

$$L_k = (1+(0.5\delta\beta h)^2)(\beta^{2k}-(\beta-2\alpha)^{2k}) + \beta^{2k}\, 0.5\alpha(\alpha-\beta)\delta^2h^2 .$$

Since $L_2 = \alpha(\alpha-\beta)(-1.5\delta^2h^2\beta^4 - 8\beta^2 - 4(4+\delta^2\beta^2h^2)\alpha(\alpha-\beta) < 0$

and $\qquad L_k = \beta^2 L_k - \alpha(\alpha-\beta)(4+(\delta\beta h)^2)(\beta-2\alpha)^{2k}$

we conclude that $L_k < 0$, $k=2,3,\ldots$ and that $T \leq 0$.

(iii) Introducing the notation $t = 0.5\delta\beta h$, $y(t) = t^2\sinh^{-2}t$ from (ii) we have $y(t) \leq P \leq 1 + t^2$. Since $y(t) \leq 1$ and $1-y(t) \leq t^2$ we obtain

$$|1-P| \leq \max\,(t^2, 1-y(t)) = t^2 = (0.5\delta\beta h)^2 .$$

In the case $i > m$ we have $-\beta \geq -(-\alpha) \geq 1$, $S < 0$, $S\beta > 0$, $S(\alpha,\beta) = -S(-\alpha,-\beta)$, $P(-\alpha,-\beta) = P(\alpha,\beta)$ so that we obtain (i), (ii) and (iii) as in the case $i \leq m$.

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REZIME

UNIFORMNO KONVERGENTNA ŠEMA SA KVAZIKONSTANTNIM FITING FAKTORIMA

U radu se posmatra problem (BVP) pod pretpostavkama : $f,g \in C^1[0,1]$, $g(x) \geq \gamma > 0$, $x \in [0,1]$, $\varepsilon_o \geq \varepsilon > 0$, $\varepsilon_o, \varepsilon, \gamma, A, B \in R$. Jedinstveno rešenje $u \in C^3[0,1]$ ima u opštem slučaju fenomen graničnog sloja u x=0 i x=1. Da bi što više tačaka bilo u graničnim slojevima za fiksan ukupan broj tačaka mreže diskretizacije, koristi se neekvidistantna mreža. Konvergencija uniformna po ε dobijenog diferencnog postupka dokazana je u teoremi pod istim pretpostavkama kao odgovarajuća teorema za ekvidistantnu mrežu iz [2].

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SOME FINITE-DIFFERENCE SCHEMES FOR A SINGULAR PERTURBATION PROBLEM ON A NON-UNIFORM MESH*

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1. INTRODUCTION

In paper |1| Bahvalov suggested a finite-difference method on a non-uniform mesh for solving boundary value problem (1). The mesh points in that method are completely determined as values of function $\lambda(s)$ at equidistant points, in such a way that the consistency error is uniform in perturbation parametat ε. A second order convergence uniform in ε is achived.

In the paper presented here we shall consider meshes formed with more freedom in choosing mesh pointes, yet, implying convergence uniform in ε. This enables us to achieve a greater number of mesh points in narrow regions of boundary layers.

Consider the problem

$$\begin{aligned} &Lx(t) := -\varepsilon^2 x''(t) + b^2(t)x(t) = f(t), \quad t \in (0,1) \\ (1) \qquad &x(o) = X_o, \quad x(1) = X_1 \\ &b(t) \geq \beta > o, \quad t \in [0,1] \end{aligned}$$

where $o < \varepsilon \leq \varepsilon_o, \varepsilon, \varepsilon_o, X_o, X_1, \beta \in \mathbb{R}$. If $b, f \in C[0,1]$, then there exists a unique solution $x \in C^2[0,1]$ to problem (1), |7|.

Further on, C will denote each positive constant independent of ε.

For problem (1) we can state (|1|,|4|,|8|):

LEMMA. *Let* L *be the operator in (1) and* y *any smooth function such that* $|(Ly(t))^{(j)}| \leq C$, $j=0,1,\ldots,j_o$, $t \in [0,1]$.

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Then for $j=0,1,\ldots,j_o+2$ *and* $t\in[0,1]$

(2) $|y^{(j)}(t)|\le C(1+\varepsilon^{-j}\exp(-tb(0)/\varepsilon)+\varepsilon^{-j}\exp(-(1-t)b(1)/\varepsilon)).$

If, moreover, $b^2(0)y(0)=Ly(0)$ and $b^2(1)y(1)=Ly(1)$, then

(2a) $|y^{(j)}(t)|\le C(1+\varepsilon^{2-j}\exp(-tb(0)/\varepsilon)+\varepsilon^{2-j}\exp(-(1-t)b(1)/\varepsilon)),$
$j=0,1,\ldots,j_o+2 \quad t\in[0,1].$

In this paper we shall use Bahvalov's function $\lambda(s)$, |1|. Here we shall give its definition and some properties.

Let $a>o$ and $q\in(o,o.5)$ be arbitrary constants independent of ε and

$\psi(s) = a\varepsilon\ln(q/(q-s)),\ s\in[o,q).$

Let $(\alpha,\psi(\alpha))$ denote the point of contact of a tangent line taking the value 0.5 at 0.5, to the curve $\psi(s)$. α is the solution of equation

$$\psi(x) = 0.5+\psi'(x)(x-0.5)\ .$$

It is easy to prove that this equation has a unique solution $\alpha\in[0,q)$ when $a\varepsilon\le q<0.5$. We can determine α from $\alpha=q-q/x_\infty$ where x_∞ is a limit of the series

$$x_o=1,\quad x_{k+1}=x_k-h(x_k)/h'(x_k),\quad k=0,\ 1,\ \ldots$$

and $\quad h(x)=\ln x-(1-0.5/q)x+1-0.5/(a\varepsilon)\ .$

Now we can define $\lambda(s)$. For $a\varepsilon>q$ we take $\lambda(s)=s$, $s\in[0,1]$ and for $a\varepsilon\le q$ let

$$\lambda(s)=\begin{cases}\psi(s), & s\in[0,\alpha]\\ \psi(\alpha)+\psi'(\alpha)(s-\alpha), & s\in[\alpha,0.5]\\ 1-\lambda(1-s), & s\in[0.5,1]\end{cases}.$$

For $\lambda(s)$ we have $\lambda(s)\ge 0$, $\lambda'(s)>0$, $s\in[0,1]$ and when $a\varepsilon\le q$

(3) $$\lambda''(s)\begin{cases}\ge 0, & s\in[0,\alpha]\\ =0, & s\in(\alpha,1-\alpha).\\ \le 0, & s\in[1-\alpha,1]\end{cases}$$

The contact point of the tangent, parallel with line $(s-q)/(1-2q)$, to the curve $\psi(s)$ is denoted by $(\alpha_1,\psi(\alpha_1))$. We have $\alpha \leq \alpha_1$ and $\alpha_1 = q - q_1\varepsilon$, $q_1 = a(1-2q)$.

By $(\alpha_2,\psi(\alpha_2))$ we denote the contact point of the tangent, parallel with line s, to the curve $\psi(s)$. Now $\alpha_2 \leq \alpha$, $\alpha_2 = = q - a\varepsilon$.

2. DISCRETISATION MESH

We form the discretisation of (1) on the mesh $I_h = \{t_o, t_1,\ldots,t_n\}$, $n = 2m$, $m \geq 2$, $m \in N$, where

$$t_o = 0 < t_1 < t_2 < \ldots < t_{m-1} < t_m = 0.5 ,$$

(4) $$t_{n-i} = 1 - t_i, \quad i = 0,1,\ldots,m-1 .$$

Since $t_o = 0$, $t_m = 0.5$ and the points t_i for $i = m+1,\ldots \ldots,n$ are determined when t_i, $i = 1,2,\ldots,m-1$, are known , we shall, when constructing the mesh I_h, give only the points t_i, $i=1,2,\ldots,m-1$. We shall consider two cases: case A and case B of mesh contruction.

For $a\varepsilon > q$ each non-uniform mesh I_h is convenient for the numerical solution of problem (1) when discretisation is given as in |1|, because we achieve convergence uniform in ε. From now on we shall consider only the case $a\varepsilon \leq q$, i.e. $\varepsilon_o = q/a$.

We shall not consider the case $b^2(0)y(0) = Ly(0)$ and $b^2(1)y(1) = Ly(1)$ when the uniform convergence can be easily proved by using (2a).

Let $\tau(\varepsilon)$ be such a function that $\tau(\varepsilon) < 0.5$ and

(5) $$\exp(-\tau(\varepsilon)/(a\varepsilon)) \leq C\varepsilon , \quad \varepsilon \in (0,\varepsilon_o],$$

for example $\tau(\varepsilon) = \varepsilon^{(k-1)/k}$ for $k \geq 2$, ε_o small enough,or $\tau(\varepsilon) = \lambda(\alpha_2)$. Define $K = s_\varepsilon/n_o$, where

$$s_\varepsilon = q(1-\exp(-\tau(\varepsilon)/(a\varepsilon))), \quad n_o \geq 2q/(1-2q), \quad n_o \in N .$$

Then $K(n_o+1) < 0.5$, $n_o + 2 \leq m$ and $s_\varepsilon < q$.

Case A.

Let $t_i = \lambda(Ki)$, $i=1,2,\ldots,n_o+1$. If $n_o+2 = m$, we have $t_{n_o+2} = 0.5$ and if $n_o+2 < m$, we take the points t_i, $i=n_o+2,\ldots$ $\ldots m-1$, arbitrarily, but with property (4).

Then, using (3), we have

$$\text{(6)} \qquad \begin{aligned} &t_{i+1}-t_i \geq K\,\lambda'(Ki) \geq t_i - t_{i-1} \;, \quad i=1,2,\ldots,n_o \;, \\ &t_{i+1}-t_i \leq K\,\lambda'(Ki) \leq t_i - t_{i-1} \;, \quad i=n-n_o,\ldots,n-1 \;. \end{aligned}$$

Case B.

For $i=1,2,\ldots,n_o+1$ we choose t_i as in case A, and for $i=n_o+2,\ldots,m$ let it be possible to choose t_i so that

$$t_{n_o+2}-t_{n_o+1} = t_{n_o+1}-t_{n_o} \;,$$

$$t_i-t_{i-1} \leq t_{i+1}-t_i \leq t_i-t_{i-p}, \quad i=n_o+2,\ldots,m-1 \;,$$

for some $p = p(i) \in \{2,3,\ldots,i-n_o\}$.

3. DISCRETISATION OF (1)

We shall have two discretisations of problem (1) for both cases, A and B, of mesh construction.

Case A.

Define

$$L_h^1 x(t_i) := a_i^1 x(t_{i-1}) + b_i^1 x(t_i) + c_i^1 x(t_{i+1}) \;, \quad i=1,2,\ldots,n-1$$

where

$$a_i^1 = \frac{-2}{(t_i-t_{i-1})(t_{i+1}-t_{i-1})} < 0 \;,$$

$$b_i^1 = \frac{2}{(t_i-t_{i-1})(t_{i+1}-t_i)} > 0 \;,$$

$$c_i^1 = \frac{-2}{(t_{i+1}-t_i)(t_{i+1}-t_{i-1})} < 0 \;.$$

If $b,f \in C^1[0,1]$, then

$$\rho_i^1 := -x''(t_i) - L_h^1 x(t_i) = (\theta_i^1 - t_i)x'''(\theta_i^2) , \tag{7}$$

with $t_{i-1} \le \theta_i^k \le t_{i+1}$, $k=1,2$, $i=1,2,\dots,n-1$, and if $b,f \in C^2[0,1]$ then

$$\rho_i^1 = \frac{1}{3}(t_{i+1}-2t_i+t_{i-1})x'''(t_i) + \frac{(t_{i+1}-t_i)^3}{12(t_{i+1}-t_{i-1})} x^{iv}(\sigma_i^2) + \tag{8}$$

$$+ \frac{(t_i-t_{i-1})^3}{12(t_{i+1}-t_{i-1})} x^{iv}(\sigma_i^1) ,$$

$t_{i-1} \le \sigma_i^1 \le t_i \le \sigma_i^2 \le t_{i+1}$, $i=1,2,\dots,n-1$.

The system of linear equations

$$\begin{aligned} & x_h(t_o) = X_o , \quad x_h(t_n) = X_1 \\ & \varepsilon^2 L_h^1 x_h(t_i) + b(t_i)x_h(t_i) = f(t_i) , \quad i=1,2,\dots,n-1 , \end{aligned} \tag{9}$$

with solution $x_h = (x_h(t_o), x_h(t_1), \dots, x_h(t_n))^T \in \mathbb{R}^{n+1}$ is a discretisation of (1).

We can write discretisation (9) in the form:

$$\varepsilon^2 A_h x_h + B_h x_h = f_h ,$$

where $B_h = \mathrm{diag}(0, b(t_1), \dots, b(t_{n-1}), 0) \in \mathbb{R}^{n+1,n+1}$,

$f_h = (X_o, f(t_1), \dots, f(t_{n-1}), X_1)^T \in \mathbb{R}^{n+1}$,

$$A_h = \begin{bmatrix} \varepsilon^{-2} & & & & & \\ a_1^1 & b_1^1 & c_1^1 & & & \\ & a_2^1 & b_2^1 & c_2^1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & a_{n-1}^1 & b_{n-1}^1 & c_{n-1}^1 \\ & & & & & \varepsilon^{-2} \end{bmatrix}$$

Elements of matrix A_h equal to zero are not indicated.

Case B.

Now we shall define the discretisation of (1) on the mesh for case B. For $i=0,1,\dots,n_o$ and $i=n-n_o,\dots,n$ the discretisation is given as in (9), and otherwise by

$$\varepsilon^2 L_h^2 x_h(t_i) + b(t_i)x_h(t_i) = f(t_i), \text{ for } i=n_o+1,\dots,m\,,$$

$$\varepsilon^2 L_h^3 x_h(t_i) + b(t_i)x_h(t_i) = f(t_i), \text{ for } i=m+1,\dots,n-n_o-1,$$

where

$$L_h^2 x(t_i) = d_i x(t_{i-p}) + c_i x(t_{i-1}) + b_i x(t_i) + a_i x(t_{i+1}),$$

$$L_h^3 x(t_i) = a_i x(t_{i-1}) + b_i x(t_i) + c_i x(t_{i+1}) + d_i x(t_{i+p}).$$

Natural numbers $p=p(i)$ for $i=n_o+2,\dots,m-1$ are given in the construction of the mesh for case B, and for $i=m+1,\dots,n-n_o-2$ we take $p(i)=p(n-i)$. Since $t_{n_o+2}-t_{n_o+1} = t_{n_o+1}-t_{n_o}$ and $t_{m+1}-t_m=t_m-t_{m-1}$ imply $d_{n_o+1}=d_m=0$, we can formally take $p(n_o+1)=p(m)=0$ and $p(n-n_o-1)=0$.

The coefficients in the given schemes are, |5|:

$$a_i = \frac{2(z_2+z_3)}{z_1(z_1-z_2)(z_1-z_3)}, \qquad b_i = \frac{-2(z_1+z_2+z_3)}{z_1 z_2 z_3},$$

$$c_i = \frac{2(z_1+z_3)}{z_2(z_2-z_1)(z_2-z_3)}, \qquad d_i = \frac{2(z_1+z_2)}{z_3(z_3-z_1)(z_3-z_2)}$$

where $z_k = z_k(i)$, $k=1,2,3$, :

$$z_1 = \begin{cases} t_{i+1}-t_i, & \text{for } L_h^2 \\ -(t_i-t_{i-1}), & \text{for } L_h^3 \end{cases} \qquad z_2 = \begin{cases} -(t_i-t_{i-1}), & \text{for } L_h^2 \\ t_{i+1}-t_i, & \text{for } L_h^3 \end{cases}$$

$$z_3 = \begin{cases} -(t_i-t_{i-p(i)}), & \text{for } L_h^2 \\ t_{i+p(i)}-t_i, & \text{for } L_h^3 \end{cases}$$

For a_i, b_i, c_i, d_i from L_h^2 we can state $a_i<0$, $b_i>0$, $c_i \le 0$, $d_i \le 0$.

For $k=2,3$ and $b,f \in C^2[0,1]$ we have

$$\rho_i^k := -x''(t_i) - L_h^k x(t_i) = \frac{1}{12}(z_1z_2+z_1z_3+z_2z_3)x^{iv}(\delta_i^k) \,, \tag{10}$$

$$t_{i-p} \le \delta_i^2 \le t_{i+1} \,, \quad t_{i-1} \le \delta_i^3 \le t_{i+p}$$

As in case A, we can now write the discretisation in the form

$$\varepsilon^2 A_h x_h + B_h x_h = f_h$$

where B_h and f_h are defined as before. The matrix A_h for this case is formed analogously as in case A.

4. CONSISTENCY ERROR

Define $r = (0, r_1, r_2, \ldots, r_{n-1}, 0)^T \in \mathbb{R}^{n+1}$ with $r_i = \varepsilon^2 \rho_i^1$, $i=1,2,\ldots,n-1$, where ρ_i^1 is given by (7) or (8), and let R denote:

$$R = (0, R_1, R_2, \ldots, R_{n-1}, 0)^T \in \mathbb{R}^{n+1} \,,$$

$$R_i = \varepsilon^2 \begin{cases} \rho_i^1 \,, & i=1,2,\ldots,n_o \\ \rho_i^2 \,, & i=n_o+1,\ldots,m \\ \rho_i^3 \,, & i=m+1,\ldots,n-n_o-1 \,, \\ \rho_i^1 \,, & i=n-n_o,\ldots,n \end{cases}$$

where ρ_i^k, $k=2,3$, are given by (10). For $x=(x_1,x_2,\ldots,x_{n+1})^T \in \mathbb{R}^{n+1}$ let $\|x\|_\infty = \max_{1\le i\le n+1} |x_i|$, and let h_A, h_B denote:

$$h_A = \max_{n_o+1 \le i \le m} (t_i - t_{i-1}) \,, \qquad h_B = \max_{n_o+2 \le i \le m-1} (t_i - t_{i-p(i)})$$

and $d = \min(b(0), b(1))$.

Case A.

THEOREM 1. *Let* $b, f \in C^1[0,1]$ *and* $a \geq 1/d$. *Then*

$$\|r\|_\infty \leq C(1/n_o + h_A) ,$$

where r *is given by* ρ_i^1 *from (7).*

P r o o f: Using (7) and (2) we have

$$|r_i| \leq \varepsilon^2 \max(t_{i+1}-t_i,\ t_i-t_{i-1}) C(1+\varepsilon^{-3}(\exp(-t_{i-1}b(0)/\varepsilon) +$$

$$+ \exp(-(1-t_{i+1})b(1)/\varepsilon))), \quad i=1,2,\dots,n-1 .$$

If $n_o+1 \leq i \leq n-n_o-1$, then

$$t_{i-1} \geq t_{n_o} = \lambda(s) = \tau(\varepsilon) \quad \text{and}$$
$$t_{i+1} \leq t_{n-n_o} = 1-t_{n_o} = 1-\tau(\varepsilon)$$

and because of (5) and $ad \geq 1$, we conclude

$$(11) \qquad |r_i| \leq C \max(t_{i+1}-t_i,\ t_i-t_{i-1}) .$$

When $1 \leq i < n_o+1$, we have $t_{i+1} < 0.5$, and $\varepsilon^{-1}\exp(-(1-t_{i+1})b(1)/\varepsilon)$ is bounded uniformly in ε. From this and because of (6) we get

$$|r_i| \leq C(t_{i+1}-t_i)(1+\varepsilon^{-1}\exp(-t_{i-1}b(0)/\varepsilon)) \leq$$
$$\leq C K \lambda'(K(i+1))(1+\varepsilon^{-1}\exp(-t_{i-1}b(0)/\varepsilon)) \leq$$
$$\leq \frac{C}{n_o} \lambda'(K(i+1))(1+\varepsilon^{-1}\exp(-t_{i-1}b(0)/\varepsilon)) .$$

Now let us show

$$(12) \qquad |r_i| \leq \frac{C}{n_o} .$$

Since

$$(13) \qquad \lambda'(s) \leq \psi'(\alpha) \leq \psi'(\alpha_1) = 1/(1-2q) ,$$

we just have to prove

$$(14) \qquad \lambda'(K(i+1))\varepsilon^{-1}\exp(-t_{i-1}b(0)/\varepsilon) \leq C .$$

1. $\alpha_2 \le K(i-1)$. Then we have $\lambda(\alpha_2) \le t_{i-1}$, and

$$\exp(-t_{i-1}b(0)/\varepsilon) \le \exp(-\lambda(\alpha_2)b(0)/\varepsilon) =$$
$$= ((q-\alpha_2)/q)^{ab(0)} = (a\varepsilon/q)^{ab(0)} ,$$

from where, using (13), we get (14), because $a\varepsilon \le q$ and $ad \ge 1$.

2. $K(i-1) < \alpha_2$. Now we consider two cases.

2.1. $K(i-1) \le \min(\alpha, q-4K)$. We conclude

$$\exp(-t_{i-1}b(0)/\varepsilon) = [(q-K(i-1))/q]^{ab(0)} \quad \text{and}$$
$$\lambda'(K(i+1)) \le \psi'(K(i+1)) = a\varepsilon/(q-K(i+1)) \le 2a\varepsilon/(q-K(i-1)) ,$$

because from $K(i-1) \le q-4K$ we get $0.5(q-K(i-1)) \le q-K(i+1)$. The above relations imply (14).

2.2. $\min(\alpha, q-4K) < K(i-1) < \alpha_2$. Since $\alpha \ge \alpha_2$, we have

$$q-4K < K(i-1) < \alpha_2 = q-a\varepsilon ,$$

$$\varepsilon < 4K/a < \frac{4q}{a}\frac{1}{n_o} . \tag{15}$$

Now consider the inequality

$$|r_i| \le 2\varepsilon^2 \max_{t\in[t_{i-1},t_{i+1}]} |x''(t)| \le C\varepsilon^2(1+\varepsilon^{-2}\exp(-t_{i-1}b(0)/\varepsilon)+$$
$$+ \varepsilon^{-2}\exp(-(1-t_{i+1})b(1)/\varepsilon)) \le C(\varepsilon^2+\exp(-t_{i-1}b(0)/\varepsilon)) .$$

Because of

$$\exp(-t_{i-1}b(0)/\varepsilon) < (4K/q)^{a\,b(0)} < (4/n_o)^{a\,b(0)}$$

and (15), we get (12) directly.

For the case $n-n_o-1 < i \le n-1$ we can prove (12) analogously .

From (11) and (12) we get the statement of the theorem.

THEOREM 2. *Let* $b, f \in C^2[0,1]$ *and* $ad \geq 2$. *Then*

$$\|r\|_\infty \leq C(1/n_o^2 + h_A) ,$$

where r *is given by* ρ_i^1 *from (8).*

P r o o f: When $n_o+1 \leq i \leq n_o-1$, we can prove relation (11) in the same way as in Theorem 1. For other values of i we shall prove

$$(16) \qquad |r_i| \leq C / n_o^2$$

Using (6) we have for $1 \leq i < n_o + 1$

$$|r_i| \leq \varepsilon^2 (P+Q),$$

$$P = \frac{1}{3} (t_{i+1} - 2t_i + t_{i-1}) |x'''(t_i)|,$$

$$Q = \frac{1}{12} (t_{i+1} - t_i)^2 2. \max_{t \in [t_{i-1}, t_{i+1}]} |x^{1V}(t)|.$$

We can show that $\varepsilon^2 Q \leq C/n_o^2$ analogously to Theorem 1.

Now let us consider P. We have

$$t_{i+1} - 2t_i + t_{i-1} \leq K^2 \lambda''(K(i+1)) \leq K^2 \psi''(\alpha_1) =$$
$$= K^2 a\varepsilon/(q-\alpha_1)^2 \leq C/(n_o^2 \varepsilon) \quad .$$

Using the above inequality we can prove that $\varepsilon^2 P \leq C/n_o^2$ when $Ki \geq \alpha_2$, in the same way as in part 1. of the proof of Theorem 1.

When $Ki < \alpha_2$ and $Ki \leq \min(\alpha, q-2K)$, we use

$$\lambda''(K(i+1)) = a\varepsilon/(q-K(i+1))^2 \leq 4a\varepsilon/(q-Ki)^2$$

to conclude the same fact, analogously to part 2.1. of the proof of Theorem 1.

When $q-2K < Ki < \alpha_2$, the proof is the same as in the previous theorem.

For $n-n_o-1 < i \leq n-1$ the proof is analogous.

Case B.

THEOREM 3. *Let* $b, f \in C^2[0,1]$ *and* $ad \geq 2$. *Then*
$\|R\|_\infty \leq C(1/n_o^2 + h_B^2)$.

P r o o f: When $i=1,2,\dots,n_o$ and $i=n-n_o,\dots,n$, we have
$$|R_i| = |r_i| \leq C/n_o^2 .$$

The proof is the same as in the previous theorem.

For other i we have, because of (10),

$$|R_i| \leq C\varepsilon^2 \max(z_2^2, z_3^2)(1+\varepsilon^{-4}\exp(-t_{i-p(i)}b(0)/\varepsilon)),\ i=n_o+1,\dots,m$$
$$|R_i| \leq C\varepsilon^2 \max(z_2^2, z_3^2)(1+\varepsilon^{-4}\exp(-(1-t_{i+p(i)})b(1)/\varepsilon)),$$
$$i=m+1,\dots,n-n_o-1 .$$

But here we have $t_{i-p(i)} \geq \tau(\varepsilon)$ and $t_{i+p(i)} \leq 1-\tau(\varepsilon)$, and from (5) and $ad \geq 2$ we conclude

$$|R_i| \leq C \max_{n_o+2\leq j\leq m-1} (t_j - t_{j-p(j)})^2,\quad n_o+1 \leq i \leq n-n_o-1.$$

This completes the proof of the theorem.

5. CONVERGENCE UNIFORM IN ε

Discretisations of (1) in both cases of paragraph 3. can be written in the form

(17) $$\varepsilon^2 A_h x_h + B_h x_h = f_h ,$$

where A_h, $B_h \in \mathbb{R}^{n+1,n+1}$, $x_h, f_h \in \mathbb{R}^{n+1}$ are defined as in paragraph 3. Let x^h denote the restriction of the exact solution of problem (1) to mesh I_h, and let $r, R \in \mathbb{R}^{n+1}$ be the same as in the previous paragraph.

Now we have

(18) $$\varepsilon^2 A_h x^h + B_h x^h = f_h - r$$

for case A and

$$\varepsilon^2 A_h x^h + B_h x^h = f_h - R \tag{19}$$

for case B.

From (17) and (18) we get

$$\varepsilon^2 A_h (x^h - x_h) + B_h (x^h - x_h) = - r \tag{20}$$

and, analogously, for case B:

$$(\varepsilon^2 A_h + B_h)(x^h - x_h) = - R . \tag{21}$$

THEOREM 4. *The matrices* $\varepsilon^2 A_h + B_h$ *in both cases* A *and* B *are regular and*

$$\|(\varepsilon^2 A_h + B_h)^{-1}\|_\infty \le 1/\min(\beta^2, 1) .$$

P r o o f: For matrix $C_h = [c_{ij}] \in \mathbb{R}^{n+1,n+1}$ defined by

$$C_h = \varepsilon^2 A_h + B_h$$

we have

$$c_{ii} > 0, \quad i = 0, 1, \ldots, n ,$$

$$c_{ij} \le 0, \quad i \ne j, \ i,j = 0,1,\ldots,n .$$

Since $a_i^1 + b_i^1 + c_i^1 = 0$, $i=1,2,\ldots,n$ and

$$a_i + b_i + c_i + d_i = 0, \quad i=1,2,\ldots,n ,$$

we get

$$\sum_{\substack{j=o\\ j\ne i}}^{n} |c_{ij}| = - \sum_{\substack{j=o\\ j\ne i}}^{n} c_{ij} = \begin{cases} \bar{b}_i & , \ i=1,2,\ldots,n-1 \\ 0 & , \ i=0,n \end{cases}$$

where $\bar{b}_i$ denotes b_i^1 or b_i. Now

$$S := \min_{o\le i\le n} \Big(|c_{ii}| - \sum_{\substack{j=o\\ j\ne i}}^{n} |c_{ij}|\Big) = \min_{1\le i\le n-1} (c_{ii} - \bar{b}_i, 1) .$$

Because of $c_{ii} = \bar{b}_i + b^2(t_i)$ and $b(t_i) \ge \beta$, $i=1,2,\ldots,n-1$, we conclude

$$S \geq \min(\beta^2, 1) > 0 \; .$$

Then Theorem 1. from |9| implies that C_h is a regular matrix and

$$\| C_h^{-1} \|_\infty \leq 1 / \min(\beta^2, 1) \; ,$$

which completes the proof.

THEOREM 5. *Let* x_A^h *and* x_B^h *denote restrictions of the exact solution of problem (1) to* I_h *in case* A *and in case* B *, respectively, and let* x_h^A *and* x_h^B *denote solutions to discretisations in case* A *and* B, *respectively. Then*

$$(22) \qquad \| x_A^h - x_h^A \|_\infty \leq C \begin{cases} 1/n_o + h_A \; , \textit{when} \;\; b,f \in C^1[0,1] \, , \; ad \geq 1 \\ 1/n_o^2 + h_A \; , \textit{when} \;\; b,f \in C^2[0,1] \, , \; ad \geq 2 \end{cases} ,$$

$$(23) \qquad \| x_B^h - x_h^B \|_\infty \leq C(1/n_o^2 + h_B^2) \; , \quad \textit{when} \;\; b,f \in C^2[0,1] \, , \; ad \geq 2.$$

P r o o f: From (20) we have

$$\| x_A^h - x_h^A \|_\infty \leq \| (A_h + B_h)^{-1} \|_\infty \; \| r \|_\infty \quad ,$$

and using the previous theorem and Theorems 1. and 2. we conclude (22).

Using Theorem 3. we prove (23) for case B.

6. REMARKS

1. We use a symmetric mesh and even n just to show the main ideas of the paper in a simpler manner.

2. Even the function $\lambda(s)$ need not be centrally symmetric. We can take $\lambda_1(s)$ for $s \in [0, 0.5]$ with $q = q_1$, $a = a_1$ and $\lambda_2(s)$ for $s \in [0.5, 1]$ with $q = q_2$, $a = a_2$, where $a_1 b(0)$, $a_2 b(1) \geq k$, $k = 1, 2$.

3. If we take $t_i = \lambda(i/n)$, we obtain Bahvalov's result |1|.

4. It is naturally more interesting to consider the case $s_\varepsilon \leq \alpha$. When we take $\tau(\varepsilon) = \lambda(\alpha_2)$, we have $s_\varepsilon = \alpha_2 \leq \alpha$.

Besides, for each ε there is $k=2,3,\ldots$, such that $s_\varepsilon \leq \alpha_2 \leq \alpha$ when we take $\tau(\varepsilon) = \varepsilon^{(k-1)/k}$.

5. The error of the numerical method gets smaller when n grows, but in such a way that n_o grows and the maximal step od difference schemes decreases outside the boundary layers.

7. NUMERICAL EXAMPLE

We shall illustrate the theoretical results with some computational results for the problem

$$-\varepsilon^2 x'' + x = 1, \quad x(0) = x(1) = 0$$

with solution

$$x(t) = 1 - \frac{\exp((t-0.5)/\varepsilon) + \exp((0.5-t)/\varepsilon)}{\exp(0.5/\varepsilon) + \exp(-0.5/\varepsilon)} .$$

This example represents a linear model of a catalytic reaction and it has been numerically treated in |2|, |6|, |8|.

Since $x(t)=x(1-t)$, we shall find the numerical solution only for $t \in [0,0.5]$. The number of mesh points in $(0,0.5]$ is denoted as before by m. The number m_s denotes how many mesh points are in $I_h \cap (0,\varepsilon]$.

In tables I, II and III we give the results for m=20, q=0.4, a=2. The values

$$g_s = \max \left\{ \left| \frac{x(t_i)-x_h(t_i)}{x_h(t_i)} \right| : t \in I_h \cap (0.\varepsilon] \right\}$$

and

$$g = \max \left\{ \left| \frac{x(t_i)-x_h(t_i)}{x_h(t_i)} \right| : t_i \in I_h \right\} ,$$

where $x(t)$ denotes the exact solution and $x_h(t)$ denotes the numerical solution to the given problem, are given in % .

Table I contains the numerical results for case A and for case B. Number k is the smallest integer which satisfies

$$\tau(\varepsilon) := \varepsilon^{(k-1)/k} \leq 1.5\,\varepsilon \ .$$

We used the meshes with points $t_i = \lambda(Ki)$, $i=1,2,\ldots,n_o+1$, and t_i, $i=n_o+2,\ldots,m-1$, which are given in Table II.

For $\varepsilon=2^{-9}$, 2^{-13}, 2^{-17} in case B it is not possible to keep the same number (nine) of mesh points in $(0,\varepsilon]$, because of the conditions that the mesh satisfy (2. Case B). Put we used the knots t_{n_o+2} and t_{n_o+3} with the property

$$\varepsilon \leq t_{n_o+2} - t_{n_o+1} \leq t_{n_o+3} - t_{n_o+2} \leq h_B \ ,$$

and a non-equidistant three-point scheme at mesh points t_{n_o+1}, t_{n_o+2} . Then estimate (23) is still valid, since for a three-point scheme we have

$$|r_i| \leq C\cdot\varepsilon \max(t_{i+1}-t_i, t_i-t_{i-1}) \leq h_B^2 \ , \ i=n_o+1, n_o+2 \ ,$$

when $ad \geq 2$.

Table III shows the numerical results for the case $t_i = \lambda(i/m)$ (as in |1|).

TABLE I

ε	k	n_o	m_s	Case A g_s%	Case A g%	Case B g_s%	Case B g%
2^{-5}	9	12	9	0.226	0.414	0.126	0.659
2^{-9}	16	12	9	0.596	1.388	2.402	3.974
2^{-13}	23	12	9	1.033	2.200	4.868	7.975
2^{-17}	30	12	9	1.385	2.232	7.548	12.175

TABLE II, Case A

	$\varepsilon=2^{-5}$	$\varepsilon=2^{-9}$	$\varepsilon=2^{-13}$	$\varepsilon=2^{-17}$
t_{14}	5.97 E-2	3.71 E-3	2.51 E-4	1.42 E-5
t_{15}	7.30 E-2	5.08 E-3	4.61 E-4	2.61 E-5
t_{16}	9.57 E-2	9.17 E-3	1.41 E-3	1.23 E-4
t_{17}	1.34 E-1	2.14 E-2	5.68 E-3	9.19 E-4
t_{18}	2.00 E-1	5.82 E-2	2.49 E-2	7.45 E-3
t_{19}	3.11 E-1	1.69 E-1	1.11 E-1	6.10 E-2

TABLE II, Case B

	$\varepsilon=2^{-5}$	$\varepsilon=2^{-9}$	$\varepsilon=2^{-13}$	$\varepsilon=2^{-17}$
t_{14}	5.78 E-2	7.16 E-3	6.92 E-4	7.38 E-5
t_{15}	6.85 E-2	2.26 E-2	1.63 E-2	1.57 E-2
t_{16}	8.77 E-2	3.80 E-2	3.19 E-2	3.13 E-2
t_{17}	1.22 E-1	6.88 E-2	6.31 E-2	6.26 E-2
t_{18}	1.85 E-1	1.30 E-1	1.26 E-1	1.25 E-1
t_{19}	2.97 E-1	2.54 E-1	2.50 E-1	2.50 E-1

TABLE III

ε	m_s	g_s%	g%
2^{-5}	6	0.130	0.119
2^{-9}	6	0.132	0.220
2^{-13}	6	0.134	0.297
2^{-17}	6	0.135	0.389

From the numerical results we can conclude that g_s and g do not change much when ε decreases, i.e. the convergence of given difference schemes is uniform in ε. The results from Table III are determined by ε, q, a and m. Table I shows that it is possible to achieve a greater number m_s, than in

Bahvalov's case. However, g_s and g are greater, but still tolerable, especially g_s - which is more interesting.

These are only some possibilities of mesh construction. The automatical construction of an optimal mesh has not been considered.

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REZIME

NEKE DIFERENCNE ŠEME ZA SINGULARNI PERTURBACIONI PROBLEM NA NEEKVIDISTANTNOJ MREŽI

U radu se posmatra diskretizacija problema (1) na neekvidistantnoj mreži (4). Pri tom se mreža I_h formira tako da diskretni analogoni za (1), dobijeni primenom operatora L_h^1, L_h^2, i L_h^3 , slučaj A i slučaj B, imaju jedinstvena rešenja koja uniformno po ε konvergiraju ka rešenju problema (1) kada $n_o \to \infty$ i $h_A \to 0$, odnosno $h_B \to 0$. Kada se čvorovi t_i mreže I_h odredjuju prema $t_i = \lambda(i/n)$, $i=0,1,\dots,n$, dobija se poznati rezultat Bahvalova |1|. Slobodniji izbor čvorova mreže koji se predlaže u ovom radu omogućava da, u odnosu na mrežu Bahvalova, veći broj čvorova leži u uskom graničnom sloju, pri istom ukupnom broju tačaka mreže.

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ON ALTERNATIVE NONSTATIONARY ITERATIVE PROCEDURES

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In this paper the equation

(1) $x = Tx + f \ , \ f \in B$

is considered in a Banach´s partially ordered space B. It is supposed that T is a linear operator and that

(2) $u \leq v \Rightarrow Tu \geq Tv \ , \ (u, v \in B)$.

To solve equation (1) a nonstationary iterative procedure is used

(3) $$\begin{cases} z_n = T_n z_{n-1} + f & (n=1,2,\ldots, \text{ or } n=2,3,\ldots) \\ T_n x = Tx + \rho_n \ , & \rho_n \in B, \quad (z_o, z_1 \in B) \ . \end{cases}$$

"Auxiliary" operators A_k and B_k are introduced so that

$$B_k x \leq \rho_k \leq A_k x$$

where x is a fixed element of space B which is expressed via sequence (3). In this way it is possible to construct invariant intervals for operator T´.

(4) $T'x = Tx + f$

which allows an a posteriori error estimation and an acceleration of the procedure (3). The achieved results represent a generalization of the results from |1| concering the alternative

iterative sequences. In Lemma1 the idea of proving the statement 1.2 |1| was used. For the sequence z_k it cannot be stated that it converges to the solution of equation (1), but for each k, fixed by the conditions of the following theorems, it is possible to determine a neighbourhood of the point x^*, x^*- being the solution of equation (1), to which z_k belongs. Furthermore, it is possible to determine the point from that neighbourhood which represents a better approximation for x^* than z_k is. The neighbourhood diametar depends on the operators A_k and B_k and can be made small up to the extent to which we are able to determine these operators. Let us list some notations which will be used in this paper.

$$x_n = Tx_{n-1} + f,\ x_o \in B \quad (n=1,2,\dots) \tag{5}$$

$$\begin{cases} \delta z_{2i} = z_{2i} - z_{2i-1} & (i=1,2,\dots) \\ \delta z_{2i+1} = z_{2i} - z_{2i+1} & (i=0,1,\dots) \end{cases} \tag{6}$$

$$Gz_k(s) = z_k + s\delta z_k \tag{7}$$

$$Iz_k(s,t,p,q) = [Gz_k(s) + t_k\ ,\ Gz_k(p) + q_k] \tag{8}$$

$$\begin{cases} \Phi_k(A,B) = -T_{k-1}A_k z_{k-2} + B_k z_{k-1} + (B_kA_k + B_k)u_k - B_kA_kz_{k-2} \\ \Phi_k'(A,B) = -T_{k-1}B_kz_{k-2} + A_kz_{k-1} + (A_k^2 + B_k)u_k + A_kB_{k-2} \end{cases} \tag{9}$$

where

$$\begin{cases} u_k = z_{k-1} - (A_k+B_k)z_{k-2} & \text{for } k = 2i \\ u_k = z_{k-2} & \text{for } k = 2i+1 \end{cases} \tag{10}$$

$$\begin{cases} \phi_k(A,B) = (B_k - A_k)z_{k-1} + (B_kA_k + A_kB_k)u_k - B_kA_kz_{k-2} + A_k f \\ \phi_k'(A,B) = (A_k - B_k)z_{k-1} + (A_k^2 + B_k^2)u_k + A_kB_kz_{k-2} + B_k f \end{cases} \tag{11}$$

LEMMA 1. *Let the equation (1) be given in* B *with linear monotonously non-increasing operator* T *which satisfies condition (2). Let for some* j *in sequence (5) there exist* $p_i, q_i \in B$ *so that it holds that*

(12) $\quad p_i \leq \delta x_i \leq q_i \quad (i=j-1,j)$ *and*

(13) $$\begin{cases} p_j \geq s_j q_{j-1} + S_j q_j \\ q_j \leq S_j p_{j-1} + s_j p_j \end{cases}$$

where s_j *and* S_j *are real numbers and*

(14) $$0 < s_j \leq S_j$$

Then, for $j=2i$ *operator* T' *maps the interval* $Ix_j(-S,0,-s,0)$ *into itself and for* $j=2i+1$ *the same operator maps the interval* $Ix_j(s,0,S,0)$ *into itself.*

P r o o f. Let $j=2i$. From (12) and (13) we get

(15) $$s_j\delta x_{j-1} + S_j\delta x_j \leq \delta x_j \leq S_j\delta x_{j-1} + s_j\delta x_j$$

By applying operator T onto the inequality (15), because of $\delta x_j = -T\delta x_{j-1}$, we get

$$x_j - s_j\delta x_j \geq T(x_j - S_j\delta x_j) + f$$

$$v_o \geq Tu_o + f = v_1$$

$$v_o = x_j - S_j\delta x_j$$

$$u_o = x_j - s_j\delta x_j$$

From (15) it also follows that

$$u_o \leq Tv_o + f = u_1$$

Because of (14) $u_o \leq v_o$ so that

$$u_o \leq u_1 \leq v_1 \leq v_o$$

For an arbitrary $s \in [u_o, v_o]$ it holds that

$$u_1 = Tv_o + f \leq Ts + f \leq Tu_o + f = v_1$$

Hence, operator T' leaves the interval $Ix_j(-S,0,-s,0)$ invariant. In a similar way, the statement for $j=2i+1$ can be proved.

COROLLARY 1. *The quantities* p_i, q_i $(i=j,j-1)$ *determined in Lemma 1 are nonnegative.*

THEOREM 1. *Let the linear monotonously nonincreasing operator* T *be defined in* B. *Let for some* $k \geq 2$ *in sequence (3) the following hold*

1.1. $(T-A_k)u_k \leq T_n u_k \leq (T+B_k)u_k \qquad (n=k,k-1)$

$u_k = z_{k-2},\ z_{k-1} - (A_k+B_k)z_{k-2}$

1.2. There are $g_k, G_k \in R$ *such that*

$0 < g_k \leq G_k$

a). for $k=2i$

$$(1-G_k)\delta z_k - g_k\delta z_{k-1} \geq (A_k+g_kB_k+G_kB_k)w_k + \Phi_k(A,B) + G_k\cdot\Phi'_k(A,B)$$

$$G_k\delta z_{k-1} - (1-g_k)\delta z_k \geq (B_k+g_kA_k+G_kA_k)w_k + \Phi'_k(A,B) + g_k\Phi_k(A,B)$$

$$w_k = z_{k-1} - (A_k+B_k)z_{k-2}$$

b) $k=2i+1$

$$(1-G_k)\delta z_k - g_k\delta z_{k-1} \geq (B_k+g_kA_k+G_kA_k)z_{k-2} + \Phi'_k(A,B) + G_k\Phi_k(A,B)$$

$$G_k\delta z_{k-1} - (1-g_k)\delta z_k \geq (A_k+g_kB_k+G_kB_k)z_{k-2} + \Phi_k(A,B) + g_k\Phi'_k(A,B)$$

Then, in the case $k=2i$, *operator* T' *maps the interval* $Ix_2(-S,0,-s,0)$ *into itself and, when* $k=2i+1$, *operator* T' *maps the interval* $Ix_2(s,0,S,0)$ *into itself. In addition* $s_2=g_k$, $S_2=G_k$, *and* x_n *is determined by (5) for* $x_o=z_{k-2}$.

P r o o f. For $k=2i$ we shall determine the quantites p_i, q_i $(i=1,2)$ which satisfy the conditions of Lemma 1 for x_n determined in the above way. Let us introduce the notations

$$(16)\qquad \begin{cases} \bar{z}_n = T_{k+n-2}\bar{z}_{n-1} + f, \quad \bar{z}_o = z_{k-2} \quad (n=1,2,\dots) \\ \ell_n = (T+(-1)^n(A_k)^{2-n}(B_k)^{n-1})\ell_{n-1} + f,\ \ell_o = \bar{z}_o \\ y_n = (T+(-1)^{n-1}(A_k)^{n-1}(B_k)^{2-n})y_{n-1} + f\ ,\ y_o = \bar{z}_o \\ x_n = Tx_{n-1} + f \qquad\qquad ,\ x_o = z_{k-2} \end{cases}$$

$(A_k)^o = (B_k)^o = E$, E the identity operator.

According to condition 1.1

(17) $$\ell_1 \le \bar{z}_1 \le y_1$$

Let us show that

(18) $$\ell_2 \ge \bar{z}_2 \ge y_2$$

Because of 1,1 it follows that

$$\ell_1 = T\bar{z}_o + f + \rho_{k-1} - \rho_{k-1} - A_k\bar{z}_o \ge w_k$$

$$\ell_2 \ge T\ell_1 + B_k w_k + f \ge T\ell_1 + \rho_{k-1} + f \ge \bar{z}_2$$

$$\bar{z}_2 \ge T\bar{z}_1 - A_k\bar{z}_1 + f \ge (T-A_k)y_1 + f = y_2$$

On the basis of the above inequalities we get the following relation between x_n and $\bar{z}_k$

(19) $$\begin{cases} x_n = \bar{z}_n + (-1)^{n-1} a_n + (-1)^{n} k_n & (n=1,2\ ) \\ x_n = \bar{z}_n + (-1)^{n} b_n + (-1)^{n-1} c_n & (n=1,2\ ) \end{cases}$$

where

$$\begin{cases} x_n = y_n + (-1)^n k_n \\ \bar{z}_n = \ell_n + (-1)^{n-1} c_n \\ y_n = \bar{z}_n + (-1)^{n-1} a_n \\ \ell_n = \bar{z}_n + (-1)^n b_n \end{cases}$$

In view of (17) and (18) it holds that $a_i \ge 0$, $b_i \ge 0$, $(i=1,2\ )$. Since

(20) $$\begin{cases} T\bar{z}_o \le \bar{z}_1 - f + A_k w_k \\ -TH\bar{z}_o \le -T_{k-1}H\bar{z}_o + B_k w_k & (H=A_k,\ B_k) \end{cases}$$

(21) $$\begin{cases} c_2 = B_k T\bar{z}_o - TA_k\bar{z}_o - B_k A_k\bar{z}_o + B_k f \\ k_2 = A_k T\bar{z}_o - TB_k\bar{z}_o + A_k B_k\bar{z}_o + A_k f \end{cases}$$

after introducing (20) into (21) we get

$$(22) \qquad \begin{cases} c_2 \le \Phi_k(A,B), \\ k_2 \le \Phi_k'(A,B) \end{cases}$$

From (19), (22), (6) it follows that the quatities

$$(23) \qquad \begin{cases} p_1 = \delta z_{k-1} - A_k w_k \\ p_2 = \delta z_k - A_k w_k - \Phi_k(A,B) \\ q_1 = \delta z_{k-1} + B_k w_k \\ q_2 = \delta z_k + B_k w_k + \Phi_k'(A,B) \end{cases}$$

satisfy inequality (12) for $j=2$, while x_j is determined by (16).

The quantities (23) also satisfy inequalities (13) for $S_2=G_k$, $s_2=g_k$, hence after applying Lemma 1 the statement is proved.

In a similar way, it can be shown that for $k=2i+1$ the quantities

$$(24) \qquad \begin{cases} p_1 = \delta z_{k-1} - B_k z_{k-2} \\ p_2 = \delta z_k - B_k z_{k-2} - \Phi_k'(A,B) \\ q_1 = \delta z_{k-1} + A_k z_{k-2} \\ q_2 = \delta z_k + A_k z_{k-2} + \Phi_k(A,B) \end{cases}$$

satisfy inequalities (12) and (13) for $j=2$ and $S_2=G_k$ $s_2=g_k$.

COROLLARY 2. *Let condition 1.1 of Theorem 1 hold. Let* $Hu_k \ge 0$ $(H=A_k,B_k)$, *where* u_k *is determined by (10). Then,*

$$\Phi_k(A,B) \ge 0 \qquad \text{and} \qquad \Phi_k'(A,B) \ge 0 .$$

THEOREM 2. *Let the operators* A_k *and* B_k *be commutative with operator* T. *Theorem 1 is valid if* Φ_k *is replaced by* ϕ_k *and* Φ_k' *by* ϕ_k'.

P r o o f. The theorem can be proved in the same way as Theorem 1 but the following majorizations have to be used

$$c_2 \leq \phi_k(A,B)$$
$$k_2 \leq \phi_k'(A,B)$$

THEOREM 3. *Let a linear monotonously nonincreasing operator* T *be defined in* B. *Let for a* $k \geq 2$ *in sequence (3) it holds that*

3.1. $(T+A_k)u_k \leq T_n u_k \leq (T-B_k)u_k \quad (n=k,k-1)$

$$u_k = (z_{k-2},\ z_{k-1} - (A_k+B_k)z_{k-2})$$

3.2. There exist such real numbers g_k *and* G_k *that*

$$0 < g_k \leq G_k \qquad \textit{and}$$

a) for $k=2i$

$$(1-G_k)\delta z_k - g_k\delta z_{k-1} \geq -(A_k+G_kB_k+g_kB_k)w_k + \phi_k(B,A) - g_k\phi_k'(B,A)$$
$$G_k\delta z_{k-1} - (1-g_k)\delta z_k \geq -(B_k+g_kA_k+G_kA_k)w_k - \phi_k(B,A) - g_k\phi_k'(B,A)$$

b) for $k=2i+1$

$$(1-G_k)\delta z_k - g_k\delta z_{k-1} \geq -(A_k+G_kB_k+g_kB_k)z_{k-2} - \phi_k'(B,A) - G_k\phi_k(B,A)$$
$$(1-g_k)\delta z_k - G_k\delta z_{k-1} \leq (B_k+G_kA_k+g_kA_k)z_{k-2} + g_k\phi_k'(B,A) + \phi_k(B,A)$$

Then, operator T' *leaves invariant the interval* $Ix_2(-S,0,-s,0)$ *for* $k=2i$, *and interval* $Ix_2(s,0,S,0)$ *for* $k=2i+1$, *respectively. Here* $S_2=G_k$, $s_2=g_k$ *and* x_n *is determined by (16).*

P r o o f. The proof is similar to that for Theorem 1 where sequences ℓ_n and y_n are defined in the following way

$$\ell_n = (T+(-1)^{n-1}(A_k)^{2-n}(B_k)^{n-1})\ell_{n-1}+f, \quad \ell_o=z_{k-2} \quad (n=1,2)$$
$$y_n = (T+(-1)^n(A_k)^{n-1}(B_k)^{2-n})y_{n-1}+f, \quad y_o=z_{k-2}\ .$$

Afterwards it is necessary to show that for $k=2i$ the quantities

$$p_1 = \delta z_{k-1} + A_kw_k$$
$$q_2 = \delta z_{k-1} - B_kw_k$$

$$p_2 = \delta z_k - \Phi_k(B,A) - B_k w_k$$
$$q_2 = \delta z_k + \Phi_k'(B,A) + A_k w_k$$

and, for k=2i+1 the quantities

$$p_1 = \delta z_{k-1} + B_k z_{k-2}$$
$$q_1 = \delta z_{k-1} - A_k z_{k-2}$$
$$p_2 = \delta z_k + B_k z_{k-2} + \Phi_k(B,A)$$
$$q_2 = \delta z_k - A_k z_{k-2} - \Phi_k'(B,A)$$

satisfy inequalities (12) and (13) for $j=2$, $s_2=\sigma_k$, $S_2=G_k$.

THEOREM 4. *Let the operators* A_k *and* B_k *be commutative with operator* T. *Then, Theorem 3 holds if* Φ_k *is replaced by* ϕ_k *and* Φ_k' *by* ϕ_k'.

COROLLARY 3. *If condition 3.1 is satisfied by* $A_k u_k \leq 0$ *and* $B_k u_k \leq 0$, u_k *is determined by (10), then,*

$$\Phi_k(B,A) \leq 0 \qquad \textit{and} \qquad \Phi_k'(B,A) \leq 0.$$

In the above theorems the invariant intervals for operator T′ were determined. The interval boundaries were expressed in terms of a function of x_n and z_n. Using the relation between x_n and z_n we shall determine a somewhat wider interval $Iz_k(U,m,V,n)$, so that

$$Ix_2\ (u,0,v,0) \subset Iz_k(U,m,V,n) = Iz_k$$

We shall show that for k=2i

$$\begin{cases} m_k = -c + U_k(k+q_k') \\ n_k = k - V_k(c+p_k') \end{cases}$$

and for k=2i+1

$$\begin{cases} m_k = -c + U_k\ (-k-p_k') \\ n_k = k + V_k\ (c+q_k'), \text{ where} \end{cases}$$

$$U_k = u_2\ ,\quad V_k = v_2\ ,$$

$$(25)\qquad \begin{cases} p_k' = \delta z_{k-1} - p_1 \\ q_k' = q_1 - \delta z_{k-1} \end{cases}$$

$$\begin{cases} c_2 \le c \\ k_2 \le k \end{cases} .$$

The functions which we can introduce for c and k as well as for p_1 and q_1 differ from the theorem to theorem and should be determined for each theorem separately.

For $k=2i$ we have $U_k < 0$, $V_k < 0$ (Lemma 1.) and

$$\bar{z}_1 - q_k' \le x_1 \le \bar{z}_1 + p_k'$$
$$\bar{z}_2 - c \le x_2 \le \bar{z}_2 + k ,$$

so that

$$x_2 + U_k \delta x_2 \ge z_k - c + U_k(\delta z_k + k + q_k') = Gz_k(U) + m_k$$
$$x_2 + V_k \delta x_2 \le z_k + k + V_k(\delta z_k - c - p_k') = Gz_k(V) + n_k .$$

For $k=2i+1$ we have $U_k > 0$, $V_k > 0$ and

$$\bar{z}_1 - p_k' \le x_1 \le \bar{z}_1 + q_k'$$
$$\bar{z}_2 - c \le x_2 \le \bar{z}_2 + k ,$$

so that

$$x_2 + V_k \delta x_2 \le Gz_k(V) + n_k$$
$$x_2 + U_k \delta x_2 \ge Gz_k(U) + m_k .$$

THEOREM 5. *Let us suppose that by means of Theorem 1 or Theorem 3 the interval* Iz_k *and* Iz_{k+1} *are determined, and that*

5.1. $g_k \le g_{k+1}$, $G_k \ge G_{k+1}$

5.2. $p_i' \ge 0$, $q_i' \ge 0$, $(i=k,k+1)$, *where* p_i' *and* q_i' *are determined by (25). Then,*

$$(26)\qquad Iz_k \subseteq Iz_{k+1}$$

P r o o f. We shall demonstrate only a part of the statement concerning Theorem 1. Theorem 3 can be proved in an analogous way. According to (23) and (24) for $k=2i$

$$A_k w_k \geq 0, \quad B_k w_k \geq 0, \quad A_{k+1} z_{k-1} \geq 0, \quad B_{k+1} z_{k-1} \geq 0 ,$$

hence because of Corollary 2 $\Phi_k'(A,B) \geq 0$ and $\Phi_k(A,B) \geq 0$.

Consider the difference between the lower limits of the intervals Iz_{k+1} and Iz_k :

$$z_{k+1}+g_{k+1}\delta z_{k+1}+m_{k+1}-z_k+G_k\delta z_k-m_k \geq -\delta z_{k+1}(1-g_{k+1})+G_{k+1}\delta z_k +$$
$$+ m_{k+1}-m_k = -\delta z_{k+1}(1-g_{k+1})+G_{k+1}\delta z_k-\Phi_{k+1}(A,B)-g_k(\Phi_{k+1}'(A,B) +$$
$$+ B_{k+1}z_{k+1})+\Phi_k(A,B)+G_k(\Phi_k'(A,B)+B_k w_k) \geq G_{k+1}\delta z_k-\delta z_{k+1}(1-g_{k+1}) -$$
$$- \Phi_{k+1}(A,B)-g_{k+1}(\Phi_{k+1}'(A,B)+B_{k+1}z_{k+1})+\Phi_k(A,B)+G_k(\Phi_k'+B_k w_k) \geq 0,$$

since, according to 1.2 b) for $k+1=2i+1$

$$G_{k+1}\delta z_k-(1-g_{k+1})\delta z_{k+1}-g_{k+1}(B_{k+1}z_{k+1}+\Phi_k'(A,B)) -$$
$$- \Phi_k(A,B) \geq A_{k+1}z_{k+1}+G_{k+1}B_{k+1}z_{k+1} \geq 0 .$$

For the upper limits

$$z_k-g_k\delta z_k+n_k-z_{k+1}-G_{k+1}\delta z_{k+1}-n_{k+1} \geq (1-G_{k+1})\delta z_{k+1} -$$
$$- g_{k+1}\delta z_k-G_{k+1}\Phi_{k+1}(A,B)-G_{k+1}A_{k+1}z_{k-1}-\Phi_k'(A,B) +$$
$$+ \Phi_k'(A,B) + g_k(\Phi_k(A,B)+A_k w_k) \geq 0 .$$

In this way we have obtained relation (26) for $k=2i$. The procedure is analogous for $k=2i+1$.

Relation (26) for the intervals determined by Theorems 2 and 4 can be obtained in a similar way, if it can be stated that k and c are nonnegative.

REMARK 1. When $A_k = B_k = 0$ and $B = R^n$, Theorem 1,2,3 and 4 are reduced to Theorem 1.2 |1|.

Experimental results. In |4| and |5| a nonstationary iterative procedure for solving the Fredholm integral equation of the second kind was described.

$$u(s) = \int_a^b K(s,t)u(t)dt + f(s), \quad (s,t \in [a,b] = I) \tag{27}$$

$$f(s) \in C(I) \;, \quad K(s,t) \in C(I \times I), \quad (a,b \in R).$$

The approximate solution $z_k(s)$ is determined from the formulas

$$\begin{cases} z_o = f \\ z_1 = K_{m_1} z_o + f \\ z_k = K_{m_k} p_n r_n z_{k-1} + f, \quad (k=2,3,\dots) \end{cases} \tag{28}$$

where

a) r_n - operator, $r_n : C = C(I) \to R^n$, $r_n u = \{u(s_i)\}_{i=1}^n$

b) p_n - operator, $p_n : R^n \to C$, $p_n z = S_\Delta(z, s_j)$,

$S_\Delta(z,s_j)$ - third degree spline on the grid

$\Delta : a = s_1 < s_2 < \dots < s_n = b$

c) $K : C \to C$

$Ku = \int_a^b K(s,t)u(t)dt$

The operators K_m approximate the integral operator K and they arise as a result of replacement of the integral by some quadratic formula.

d) $K_m : C \to C$

$K_m u = \sum_{j=o}^{p} d_j(m) K(s,t_j) u(t_j)$, $p = 2^m + 1$ where $d_j(m)$ - weighting coefficient of the applied quadratic formula and m is determined by

$$\| r_n (K_m u - K_{m-1} u) \| \le \varepsilon \;, \quad \varepsilon > 0 \text{ given.}$$

The iterative procedure (28) has the from (3) if it is taken that

$$\rho_k = K_{m_k} p_n r_n z_{k-1} - K z_{k-1} .$$

In |4| and |5| the conditions were given which enable us to get the estimation of the form

(29) $\quad |\rho_k| \leq \varepsilon_1 \ , \ \varepsilon_1 = \varepsilon_1(\varepsilon, \|\Delta\| , \ m_k, \ q) \ ,$

where q is the order accuracy with which the calculations were caried out. If an estimation of the form (29) is possible, than Theorems 2 and 4 can be applied onto (28) where it has been taken that

(30) $\quad A_k = B_k = (\varepsilon_1 \ / \ (\min_t z_{k-2}(t))^{-1}) E .$

An application of Theorem 2 is demonstrated on Love´s equation

$$K(s,t) = \frac{-1}{\pi(1+(s-t)^2)} \ , \quad -1 \leq s,t \leq 1$$

$$f(s) = 1$$

The computation is stopped when for some p

$$|r_n \ (Z_p - z_{p-1})| \leq \delta \qquad .$$

The extremes were determined by means of the cubic spline so that an error of the order $0(\|\Delta\|^3)$ has been introduced (|4|).

The results are shown in Table 1. DG_k denotes the lower interval limit obtained after the k-th integration and GG_k- the upper limit, while AS_k denotes the arithmetic mean of the obtained interval. For a comparison, we give the results of Brakhage (Über die numeriche Behandlung von Integralgleichungen nach der Quadraturformel Methode, Numer.Math. 2, 183-196, (1960)).

s_i	$\tilde{u}(s_i)$
0	0.6574172
0.25	0.6638282
0.5	0.6831709
0.75	0.7148688
1	0.7577358

where $|u^*(s_i) - \tilde{u}(s_i)| \leq 0.0024$, $u^*(s)$ is the solution of equation (27).

TABLE 1.

	$\varepsilon_1 = 10^{-4}$			$\delta = 10^{-6}$		$\varepsilon = 10^{-5}$			
i	$DG_2(s_i)$	$GG_2(s_i)$	$AS_2(s_i)$	$DG_3(s_i)$	$GG_3(s_i)$	$AS_3(s_i)$	$z_3(s_i)$	$z_{18}(s_i)$	$z_2(s_i)$
1	.6566548	.6587830	.6577189	.6569388	.6578150	.6573770	.6249890	.6574104	.7288849
2	.6569104	.6569374	.6579740	.6571951	.6580708	.6576331	.6252747	.6576662	.7290754
3	.6576786	.6598005	.6587396	.6579645	.6588385	.6584015	.6261320	.6584349	.7296479
4	.6589601	.6610739	.6600170	.6592481	.6601195	.6596839	.6275620	.6597166	.7306042
5	.6607566	.6628587	.6618078	.6610472	.6619153	.6614814	.6295655	.6615236	.7319465
6	.6630678	.6651559	.6641119	.6633620	.6642258	.6637940	.6321411	.6638279	.7336788
7	.6659029	.6679728	.6669378	.6662002	.6670587	.6666296	.6352980	.6666608	.7358048
8	.6692557	.6713049	.6702805	.6695573	.6704090	.6699831	.6390278	.6700139	.7383280
9	.6731286	.6751542	.6741414	.6734343	.6742787	.6738565	.6433313	.6738865	.7412517
10	.6775193	.6795185	.6785190	.6778290	.6786654	.6782472	.6482043	.6782765	.7445774
11	.6824231	.6843927	.6834080	.6827388	.6835661	.6831524	.6536431	.6831827	.7483046
12	.6878309	.6897686	.6887999	.6881499	.6889672	.6885586	.6596296	.6885879	.7524297
13	.6937294	.6956322	.6946809	.6940510	.6948576	.6944544	.6661501	.6944826	.7569458
14	.7001002	.7019658	.7010331	.7004232	.7012186	.7008209	.6732827	.7008481	.7618415
15	.7069178	.7087445	.7078311	.7072418	.7080252	.7076335	.6806993	.7076597	.7671001
16	.7141521	.7159381	.7150452	.7144756	.7152464	.7148612	.6886642	.7148862	.7726998
17	.7217648	.7235084	.7226367	.7220736	.7228317	.7224526	.6970160	.7224901	.7786131
18	.7297127	.7314122	.7305642	.7300320	.7307768	.7304044	.7057638	.7304242	.7848067
19	.7379453	.7395997	.7387726	.7382593	.7389903	.7386248	.7147927	.7386456	.7912426
20	.7464106	.7480192	.7472150	.7467139	.7474360	.7470777	.7240670	.7470963	.7978809
21	.7550457	.7566085	.7558272	.7553496	.7560527	.7557013	.7335219	.7557197	.8046689
		g_k	0.3069177			0.3089443			
		G_k	0.3149267			0.3145148			
		$\max(GG_k(s_i)-DG_k(s_i))/2$	0.0010140			0.0004381			

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REZIME

O ALTERNATIVNIM NESTACIONARNIM ITERATIVNIM POSTUPCIMA

U |1| su date teoreme koje omogućavaju ubrzanje stacionarnih iterativnih postupaka i daju aposteriornu ocenu greške. U ovom radu su dokazane analogne teoreme za nestacionarne iterativne postupke.

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ON AN A POSTERIORI ERROR ESTIMATION IN SOLVING SOME CLASSES OF OPERATORS EQUATIONS

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In |7| and |8| some possibilities were described for determining invariant intervals for positive linear operators by means of nonstationary iterative methods. Here the application of these results will be demonstrated for approximate solutions of the systems of integral and linear equations. In this way important information about the equation solution can be obtained on the basis of two iterations alone. At the same time, an acceleration of the iterative procedure is achieved.

We shall first define some operators and list notations and theorems which will be used in the present paper. The integral operator $K : C(I) \to C(I)$, $I = [a,b]$

$$(1) \qquad Ku = \int_a^b K(s,t)u(t)dt \; , \quad u \in C(I).$$

The restriction operator $r_n : C(I) \to R^n$

$$r_n u = \{u(s_i)\}_{i=1}^n \; , \quad u \in C(I) \; .$$

The prolongation operator $P_n : R^n \to C(I)$

$$P_n z = S_\Delta(z,s) \; , \quad z \in R^n \; ,$$

where $S_\Delta(z,s)$ is a third order spline on the grid

$$\Delta : a = s_1 < s_2 < \dots < s_n = b$$

with the ordinates z_i ($i=1,2,\dots,n$).

The operators K_m and K_m' approximate the operator K and they are obtained if the integral in (1) is substituted by the Newton-Cotes quadrature formula.

$$K_m, K_m' : C(I) \to C(I)$$

$$K_m u = \sum_{j=o}^{2^m+1} d_j(m) K(s,t_j) u(t_j),$$

where m is determined by

$$\|r_n(K_m u - K_{m-1} u)\| \le \sigma \ , \ \sigma > 0 \ , \text{ is given},$$

$$K_m' u = \sum_{j=o}^{2^m+1} d_j(m) K(s,t_j) u(t_j),$$

where m is determined by

$$\|K_m' u - K_{m-1}' u\| \le \sigma \ , \quad \sigma > 0 \quad \text{is given}.$$

$$T'x = Tx + f$$

$$\delta z_k = z_k - z_{k-1}$$

$$Gz_k(s) = z_k + s\delta z_k$$

$$Iz_k(s,t,p,q) = [Gz_k(s) + t_k, \quad Gz_k(p) + q_k]$$

$$R_k(A,B) = T_{k-1} B_k z_{k-2} + B_k z_{k-1} + (B_k A_k + B_k^2 + A_k) z_{k-2}$$

$$R_k'(A) = T_{k-1} A_k z_{k-2} + A_k (z_{k-1} + z_{k-2})$$

$$r_k(A,B) = 2B_k z_{k-1} + 2B_k A_k z_{k-2} + B_k^2 z_{k-2} - B_k f$$

$$r_k'(A) = 2A_k z_{k-1} + A_k^2 z_{k-2} - A_k f$$

$$\| u \| = \max_{t \in I} |u(t)| \ , \quad u \in C(I)$$

$$\| z \| = \max_{1 \le i \le n} \|z_i\| \ , \quad z \in R^n$$

$$w_2(f,h) = \sup_{|t| \le h} \ \sup_{x-t,t,x+t \in [a,b]} |f(x+t) - 2f(x) + f(x-t)|$$

B - Banach's partially ordered space,

E - identity operator.

THEOREM 1. |8| *Let the linear positive operator* T *be defined in* B *and let for some* $k \ge 2$ *in the sequence*

$$(2)\qquad \begin{cases} z_n = T_n z_{n-1} + f\ , & (f, z_o \in B) \\ T_n x = Tx + \rho_n\ , & (\rho_n \in B) \end{cases}$$

the following hold:

There exist positive linear operators A_k and B_k such that

1.1. $(T-A_k)z_{k-2} \le T_n z_{k-2} \le (T+B_k)z_{k-2}$ $(n=k-1,k)$

1.2. $\delta z_{k-1} > (A_k+B_k)z_{k-2}$;

1.3 There exist real numbers s_k, S_k such that

a) $0 < s_k < S_k$

b) $\delta z_k - s_k(\delta z_{k-1} - \delta z_k) \ge (1+2s_k)A_k z_{k-2} + (1+s_k)r_k(A,B)$

c) $S_k(\delta z_{k-1} - \delta z_k) - \delta z_k \ge (1+2S_k)B_k z_{k-2} + (1+S_k)r_k'(A)$.

Then the operator T' maps the interval $Ix_2(\mu,0,\eta,0)$ into itself. Simultaneously $\mu_2 = s_k$, $\eta_2 = S_k$ and

$$x_n = Tx_{n-1} + f,\quad x_o = z_{k-2} \qquad (n=1,2,\dots,\ )$$

In |8| it was supposed that the value of operator T and, consequently, the sequence x_n can not be calculated exactly. For this reason by the sequence (2) the wider interval (3) is to be determined.

$$(3)\qquad Iz_k(U,a,V,b) \supseteq Ix_2(\mu,0,\eta,0)\ ,\ \text{where}$$

$$\mu_2 = U_k\ ,\quad \eta_2 = V_k\ ,$$

$$a_k = -(1+U_k)R_k(A,B) - U_k A_k z_{k-2}$$

$$b_k = (1+V_k)R_k'(A) + V_k B_k z_{k-2}$$

THEOREM 2. |8| *If the operators A_k and B_k are commutative with T, then in Theorem 1 it is possible to replace $R_k(A,B)$ with $r_k(A,B)$ and $R_k'(A)$ with $r_k'(A)$.*

THEOREM 3. |8| *Let the intervals* Iz_k *and* Iz_{k+1} *be determined by Theorem 1 and let*

3.1. $s_k \leq s_{k+1}$, $S_k \geq S_{k+1}$

3.2. $A_k z_{k-2} \geq 0$, $B_k z_{k-2} \geq 0$.

Then $Iz_{k+1} \subseteq Iz_k$

1. *We shall now consider the integral equation*

$$(4) \qquad u = Ku + f, \quad a,b \in R .$$

In order to find an approximate solution of equation (4), we shall use the nonstationary iterative procedure

$$(5) \qquad z_k = K'_{m_k} z_{k-1} + f , \quad z_o = f \quad (k=1,2,\dots).$$

THEOREM 4. *Let for solving equation (4) a nonstationary iterative procedure (5) be applied with the Newton-Cotes quadrature formula, which is exact for polynoms of the order* $p, p \leq \ell$.

Let

4.1 $K(s,t) \geq 0, \quad K(s,t) \in C^{\ell+1}(I \times I)$

4.2 $f(s) > 0, \quad f(s) \in C^{\ell+1}(I)$

4.3 $\dfrac{\partial^{\ell+1}}{\partial^{\ell+1}} \left[K(s,t) z_p(t)\right]$ $(p=k-1,k-2)$, *which does not change the sign on* I.

Let for some $k \geq 2$ *in sequence (5) it hold:*

4.4 $\delta z_{k-1} > 2\varepsilon z_{k-2}$, *where*

$$(6) \qquad \varepsilon \geq \sigma(\min_t z_{k-2}(t))^{-1}$$

4.5 $0 < m \leq M < 1$, *where*

$m = \min_t m(t,\varepsilon), \quad M = \max_t M(t,\varepsilon)$,

$m(t,\varepsilon) = (\delta z_k(t) - F(t))(\delta z_{k-1}(t) + \varepsilon z_{k-2}(t))^{-1}$

$$M(t,\varepsilon) = (\delta z_k(t) + F(t) - 2\varepsilon^2 z_{k-2}(t))\cdot(\delta z_{k-1}(t) - \varepsilon z_{k-2}(t))^{-1}$$

$$F(t) = 2\varepsilon z_{k-1}(t) + (3\varepsilon^2+\varepsilon) z_{k-2}(t) - \varepsilon f(t)$$

Then, equation (4) has a solution $u^*(s)$ *in the interval* $Iz_k(s, t, S, q)$, *where*

$$s_k = m_k/(1-m_k), \qquad S_k = M_k/(1-M_k)$$

$$t_k = -(1+s_k)(2\varepsilon z_{k-1}(t)+\varepsilon^2 z_{k-2}(t)-\varepsilon f(t))+s_k\varepsilon z_{k-2}(t)$$

$$q_k = (1+S_k)(2\varepsilon z_{k-1}(t)+3\varepsilon^2 z_{k-2}(t)-\varepsilon f(t))+S_k\varepsilon z_{k-2}(t) \ .$$

P r o o f. Since $f > 0$ then $z_{k-2} > 0$ $(k \geq 2)$. According to Theorem 3.1 |6|

$$\|Kz_i - K'_{m_{i+1}} z_i\| \leq \sigma .$$

If we put $A_k = B_k = \varepsilon E$, ε being determined by (6), then onto the iterative sequence (5) Theorem 2 can be applied from which follows the existence of an invariant interval. The existence of a solution can be obtained from the monotony of the stationary sequence formed as in Theorem 2 or from Shauder´s fixed point theorem.

For an application of Theorem 2 on the nonstationary procedure of the form

$$r_n z_1 = r_n K_{m_1} f + r_n f$$

$$r_n z_{k+1} = r_n K_{m_k} p_n r_n z_k + r_n f \ , \quad (k=1,2,\dots)$$

see |9| and |7|.

2. In |5|, the solving the system of integral equations was described

$$Y = \lambda \tilde{K} Y + F \tag{7}$$

where Y and F are vector functions

$$Y = \{y^i(t)\}_{i=1}^{n} \quad , \; F = \{f^i(t)\}_{i=1}^{n}$$

and $\tilde{K}$ a matrix $n \times n$ whose element are operators K_{ij},

$$K_{ij} : C(I) \to C(I)$$

$$K_{ij}x = \int_a^b K_{ij}(s,t)x(t)dt$$

It was assumed that $f^i(t)$ and $K_{ij}(s,t)$ are continuous and periodic functions with a period b-a for all variables. For solving system (7) an iterative procedure is to be applied

$$(8) \qquad Y_k = F + \lambda \tilde{K} S_\Delta(Y_{k-1},t) \ , \quad Y_o = F \quad (k=1,2,\ldots)$$

where

$$S_\Delta(Y_{k-1},t) = \{S_\nabla(y^i_{k-1},t\}\ _{i=1}^{n} .$$

For this a periodic cubic spline and an equidistant grid were used. It has been shown (|5|) that procedure (8) converges if

$$\frac{1}{4}\ (1+3\ \sqrt{3})\,|\lambda|\,K_o n(b-a) < 1$$

$$K_o \geq \max_{s,t} |K_{ij}(s,t)| \ , \quad (i\ ,j=1,2,\ldots,n) \ .$$

If the integral operators in (7) are positive, then Theorem 1 can be applied to procedure (8). Note that operator $K_{ij}S$ defined as

$$K_{ij}Sx = \int_a^b K_{ij}(s,t)S_\Delta(x,t)dt \ ,$$

is not linear. Let us define the operators T_k and T_{k-1} in this way:

$$(9) \qquad \begin{aligned} T_j Y &= \lambda\tilde{K}Y + \rho_j \ , \quad \rho_j = \lambda\tilde{K}R_{j-1} \quad (j=k,k-1) \\ R_j &= S_\Delta(Y_j,t) - Y_j \ . \end{aligned}$$

According to Theorem 1 in |5|, it holds that :

$$-(1+\frac{2}{3\sqrt{3}})W_2(Y_j,h) \leq R_j \leq (1+\frac{2}{3\sqrt{3}})\ W_2(Y_j,h) \ ,$$

$$W_2(Y_j,h) = \{w_2(y^i_j,h)\}\ _{i=1}^{n} \ .$$

Furthermore

$$(10) \qquad \begin{aligned} &-C\bar{K}W_2(Y_j,h) \leq \lambda\tilde{K}R_j \leq C\bar{K}W_2(Y_j,h), \text{ where} \\ &\bar{K} = \{\bar{K}_{ij}\}^n_{i,j=1} \end{aligned}$$

$$\bar{K}_{ij} = \max_{s,t} \int_a^b K_{ij}(s,t)dt, \quad C = \lambda(1+\frac{2}{3\sqrt{3}}) .$$

Before proceeding to further Theorems we shall introduce some mew notations

$$H = C\psi\ ,\ \text{a } \psi \text{ matrix } n \times n \text{ with the elements } \psi_{ij}\ ,$$

$$\psi_{ij} = \bar{K}_{ij} \max\ [w_2(y^i_{k-1},h)\ w_2(y^i_{k-1},h)]\ (\min_t |y^i_{k-2}(t)|)^{-1}$$

$$\Phi_k(t) = T_{k-1}Hy^i_{k-2}(t) + H^i y_{k-1}(t) + H(2H+E)y^i_{k-2}(t)$$

$$m^i_k(t,H) = (\delta y^i_k(t) - Hy^i_{k-2}(t) - \Phi_k(t)(\delta y^i_{k-1}(t) + Hy^i_{k-2}(t))^{-1}$$

$$M^i_k(t,H) = (\delta y^i_k(t) + Hy^i_{k-2}(t) + \Phi_k(t) - 2H^2 y^i_{k-2}(t)) \cdot (\delta y^i_{k-1}(t) - Hy^i_{k-2}(t))^{-1}$$

THEOREM 5. *Let for solving system (7) the iterative procedure (8) be applied and let the above assumptions on the continuity and periodicity of the functions* $f^i(s)$ *and* $K_{ij}(s,t)$ *hold.*

Let $K_{ij}(s,t) \geq 0,\quad f^i(s) > 0\quad (i,j=1,2,\dots,n)$

Let for somr $k \geq 2$ *in sequence (8) the following hold:*

a) $\delta y_{k-1} > Hy_{k-2}$, b) $0 < m \leq M < 1$,

$$m_k \leq \min_i\ (\min_t\ m^i_k(t,H))\ ,\qquad M_k \geq \max_i(\max_t\ M^i_k(t,H))$$

Then, system (7) has a solution within the interval

(11) $IY_k(U,a,V,b)$, *where*

$$U_k = \frac{m_k}{1-m_k},\quad V_k = \frac{M_k}{1-M_k}$$ *and* a_k *and* b_k *are defined by (3) for*

(12) $A_k = B_k = H.$

P r o o f. By a direct application of Theorem 1 the existence of an invariant interval for the operator K'

$$K'Y = \tilde{K}Y + F$$

is obtained. For this the relations (9),(10) and (12) are to be used. The solution existence is obtained by means of Schauder's fixed point theorem. Interval (11) contains invariant interval determined by Theorem 1.

In a similar way, Theorem 3 can be applied to give the relation between intervals (11) determined by steps k and k+1.

3. Let the systems of linear equations be given by

$$(13)\quad \begin{aligned} & x = Tx + f, \quad f \in R^n \\ & z = T^{\star}z + f \end{aligned}$$

T and T* are matrices $n \times n$ with elements t_{ij} and $t^{\star}_{ij}$ respectively. Let us suppose that

$$t^{\star}_{ij} = t_{ij} + r_{ij}$$

$$|r_{ij}| \leq \varepsilon$$

The following theorem gives a two-sided iterative procedure for tha approiximate solving of systems (13) via matrix T*.

THEOREM 6. *Let* $T \geq 0, T^{*} \geq 0$.

Let for some k *in the sequence*

$z_n = T^{*}z_{n-1} + f$, $(n=1,\ldots,\ )$ *the following inequalities hold*

a) $\delta z_{k-1} \geq 0$

b) $m_k \delta z_{k-1} \leq \delta z_k \leq M_k \delta z_{k-1}$

m_k *and* M_k *are real numbers such that*

$$0 < m_k \leq M_k < 1.$$

Let for some j *in the sequences*

$$(14)\quad \begin{cases} v_j = T^{*}v_{j-1} + f, & v_o = t_k + M_k \delta z_k/(1-M_k) \qquad (j=1,2,\ldots) \\ u_j = T^{*}u_{j-1} + f, & v_o = z_k + m_k \delta z_k/(1-m_k) \end{cases}$$

it hold that

$$(15)\quad \begin{cases} v_{j-1} - v_j \geq \varepsilon D v_{j-1} \\ -u_{j-1} + u_j \geq \varepsilon D' u_{j-1}, \end{cases}$$

D *matrix* $n \times n$, $d_{ij} = 1 \quad (i,j=1,2,\ldots,n)$

D′ *matrix* $n \times n$,

$$d'_{ij} = \begin{cases} d_{ij} & t^*_{ij} \geq \varepsilon \\ 0 & t^*_{ij} < \varepsilon \end{cases}$$

Then, system (13) has x^* *as a solution and the following estimation is valid*

$$u_j - \varepsilon D' u_{j-1} \leq x^* \leq v_j + \varepsilon D v_{j-1} .$$

P r o o f. If we introduce the notations

$$V_1 = (T^* + \varepsilon D)V_o + f , \quad V_o = v_{j-1}$$
$$U_1 = (T^* - \varepsilon D')V_o + f, \quad U_o = u_{j-1}$$

then, because of (15),

$$V_o - V_1 \geq 0, \quad U_1 - U_o \geq 0, \quad 0 \leq U_o \leq V_o .$$

Since

$$T^* - \varepsilon D' \leq T \leq T^* + \varepsilon D ,$$

after applying the Theorem (|3| p. 346), it follows that

$$U_1 \leq x^* \leq V_1$$

Because of (15), the statement is proved.

REMARK 1.

The conditions a) and b) determine the invariant interval for T (|2|). If such an interval could be determined in another way, then for u_o and v_o in (14) the limits of such an interval can be chosen.

REMARK 2.

The two-sided procedure (14) is performed with one matrix only, i.e. with the one for which is supposed to be given. In |3| a similar procedure is obtained for two different matrices.

The monotony is preserved while the conditions (15) resulting from the given matrix T are valid.

REMARK 3.

In Theorem 5 the errors are not taken into account either with which f was defined or the rounding errors. A control of these errors is possible if the following matrix instead of the matrix εD, is taken

$$\bar{D} = \varepsilon_j E + \varepsilon D \quad , \quad \text{where}$$

$$\varepsilon_j \geq \sigma_j \ (\min_i v^i_{j-1})^{-1}, \quad |\rho^i_j| \leq \sigma_j$$

$\rho_j = \rho'_j + r$, ρ'_j are rounding errors, and r is the error by which the vector f was given. Instead of the matrix $\varepsilon D'$, the matrix G should be taken

$$g_{ii} = \begin{cases} \varepsilon'_i + \varepsilon & \text{for } t^*_{ii} \geq \varepsilon + \varepsilon_i \\ 0 & \text{otherwise} \end{cases}$$

$$g_{ij} = \begin{cases} \varepsilon & \text{for } t^*_{ij} \geq \varepsilon, \quad i \neq j \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon'_j = \sigma_j \ (\min_i u^i_{j-1})^{-1}$$

REMARK 4.

The application of Theorem 1, 2 and 3 for solving a system of linear equations in the case when both the matrix and the vector are given by an error and when the rounding errors are present was demostrated in |7|. The ideas about the determination of operators A_k and B_k for such cases were given in |9| .

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REZIME

O APOSTERIORNOJ OCENI GREŠKE PRI REŠAVANJU NEKIH KLASA OPERATORSKIH JEDNAČINA

U |7| i |8| su date teoreme koje omogućavaju odredjivanje invarijantnog intervala za pozitivne linearne operatore primenom nestacionarnih iterativnih postupaka. Dokazi se zasnivaju na egzistenciji specijalnih pozitivnih operatora. U ovom radu je pokazana konstrukcija navedenih operatora i primena dobijenih rezultata na približno rešavanje sistema linearnih i integralnih jednačina.

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WEYL - OTSUKI SPACES OF THE SECOND AND THIRD KIND

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1. INTRODUCTION

The basic objects of spaces defined and investigate by T.Otsuki |1| are as follows: a tensor field P of the type (1,1) ($\det(P^i_j) \neq 0$) and the coefficients $'\Gamma^i_{jk}$ and $"\Gamma_j{}^i{}_k$ of the connections $'\Gamma$ and $"\Gamma$ respectivey . These connections are the contravariant respective covariant part of the regular general connection Γ , i.e. $'\Gamma$ is the ordinary affine connection with the help of which is defined the covariant derivative of a contravariant vector:

$$D_k v^i = \left(\frac{\partial v^a}{\partial x^k} + {}'\Gamma^a_{sk} v^s\right) P^i_a ;$$

$"\Gamma$ is the ordinary affine connection with help of which is defined the covariant derivative of a covariant vector:

$$D_k v_j = \left(\frac{\partial v_a}{\partial x^k} - {}"\Gamma^s_{ak} v_s\right) P^a_j ,$$

while for the tensor v^i_{jt} for example, we have:

$$D_k v^i_{jt} = \left(\frac{\partial v^a_{bc}}{\partial x^k} + {}'\Gamma^a_{sk}\, v^s_{bc} - {}"\Gamma^s_{bk} v^a_{sc} - {}"\Gamma^s_{ck} v^a_{bs}\right) P^i_a P^b_j P^c_t .$$

The connections $'\Gamma$ and $"\Gamma$ are not independent; they satisfy the condition

$$\frac{\partial P^i_j}{\partial x^k} + {}"\Gamma^i_{ak}\, P^a_j - {}'\Gamma^a_{jk}\, P^i_a = 0. \tag{1.1}$$

This cindition is equivalent with

$$D_k Q^i_j = 0 \ ,$$

where $Q = P^{-1}$, i.e.

$$P^i_j \ Q^s_j = P^s_j \ Q^i_s = \delta^i_j \ . \tag{1.2}$$

The Weyl-Otsuki space (W-O_n -space) is defined and investigated by A.Moór (|2|, |3|). This is an Otsuki space endowed with a symmetric positive definite metric tensor g_{ij} ($\det(g_{ij}) \neq 0$), and a recurrence vector γ_k such that the following conditions are satisfied:

a) the metric tensor is recurrent, i.e.

$$D_k g_{ij}(x) = \gamma_k(x) g_{ij}(x) \ ;$$

b) the covariant part "Γ of the regular general connection Γ is symmetric; and

c) (1.3) $P_{ij} = g_{is} P^s_j = g_{js} P^s_i = P_{ji}$.

In W-O_n spaces, coefficients of connection "Γ have the form |2|:

$$"\Gamma^i_{jk} = \{{}^{\,i}_{j\,k}\} - \frac{1}{2} g^{is} (\gamma_j g_{ab} Q^a_k Q^b_s + \gamma_k g_{ab} Q^a_s Q^b_j - \gamma_s g_{ab} Q^a_j Q^b_k), \tag{1.4}$$

where $\{{}^{\,i}_{j\,k}\}$ are Christoffel symbols of the second kind with respest to the tensor g_{ij}. Substituting (1.4) into (1.1) we obrain the corresponding connection ´Γ.

In this paper we investigate some differently defined Weyl-Otsuki spaces. In fact, we investigate the Otsuki space where condition c) is satisfied, and instead of conditions a) and b) - the following conditions are satisfied

a´) $D_k g_{ij}(x) = \gamma_k(x) m_{ij}(x)$,

where $m_{ij}(x)$ is a field of symmetric tensor;

b´) the contravariant part ´Γ of regular general connection is symmetric.

We have considered in |4| a special case of such $W\text{-}O_n$ spaces, namely the case $\gamma_k = 0$. Sime results obtained in |4| can be generalized for the general case a´). In fact, in exactly the same manner as in |4|, we find that the regular general connection satisfying a´), b´) and c) has the form

$$(1.5)\qquad {}''\Gamma^i_{jk} = {}''\overset{m}{\Gamma}{}^i_{jk} + \frac{1}{2} g^{si}(\gamma_q m_{pk} Q^q_s Q^p_j - \gamma_k m_{pq} Q^p_s A^q_j - \gamma_t m_{pk} Q^p_s Q^t_j)$$

$$(1.6)\qquad {}'\Gamma^i_{jk} = {}'\overset{m}{\Gamma}{}^i_{jk} + \frac{1}{2} g^{st}(\gamma_q m_{jk} Q^q_s Q^i_t - \gamma_k m_{pj} Q^p_s Q^i_t - \gamma_j m_{pk} A^p_s Q^i_t),$$

where

$$(1.7)\qquad {}''\overset{m}{\Gamma}{}^i_{jk} = \left\{ {i \atop j\ k} \right\} + \overset{o}{\nabla}_{[a} P^i_{k]} Q^a_j - \overset{o}{\nabla}_{[a} P^l_{k]} Q^{ai} g_{jl} - \overset{o}{\nabla}_{[a} P^l_{q]} Q^{ai} P_{kl} Q^q_j$$

is the metric connection, i.e. the connection with respect to which

$$D_k g_{ij} = 0,$$

while $\overset{o}{\nabla}_k$ denotes the ordinary covariant derivative with respect to $\left\{ {i \atop j\ k} \right\}$,

$$Q^{ai} = Q^a_s g^{si} = Q^i_s g^{sa} = Q^{ia},$$

and

$$(1.8)\qquad {}'\overset{m}{\Gamma}{}^i_{jk} = \left\{ {i \atop j\ k} \right\} + \overset{o}{\nabla}_{(k} P^a_{j)} Q^i_a - \overset{o}{\nabla}_{[a} P^l_{k]} Q^{at} Q^i_t P_{jl} - \overset{o}{\nabla}_{[a} P^l_{j]} Q^{at} Q^i_t P_{kl}.$$

Connection (1.8) is not a metric connection; it is only the connection satisfying, together with (1.7), condition (1.1).

We say that space satisfying conditions a´), b´) and c) is a Weyl-Otsuki space of the second kind if $m_{ij} = P_{ij}$. In the case $m_{ij} = P_{ia} P^a_j$, we say that the considered space is a Weyl-Otsuki space of the third kind.

In section 2 we investigate $W\text{-}O_n$ spaces of the second kind, and in section 3 - $W\text{-}O_n$ spaces of the third kind. In this last section (i.e. in section 3), we generalize some other results obtained in |4|.

2. WEYL-OTSUKI SPACES OF THE SECOND KIND

In this case $m_{ij} = g_{ia}P^a_j$, and connection (1.5) has the form

$$"\Gamma^i_{jk} = "\overset{m}{\Gamma}{}^i_{jk} + H^i_{jk}$$

where

$$(2.1)\qquad H^i_{jk} = -\frac{1}{2}(\gamma_k Q^i_j + \delta^i_k\tilde{\gamma}_j - \tilde{\gamma}^i g_{kj}), \quad \tilde{\gamma}_j = \gamma_a Q^a_j, \quad \tilde{\gamma}^i = \tilde{\gamma}_a g^{ai}.$$

In this section we denote: by $\overset{m}{\nabla}$ the ordinary covariant derivative with rescpect to the metric connection $"\overset{m}{\Gamma}{}^i_{jk}$ (i.e. with respect to connection (1.7), by "R the curvature tensor of the connection "Γ and by $\overset{m}{R}$ - the curvature tensor of the metric connection $"\overset{m}{\Gamma}$ i.e.:

$$"R^i_{rkj} = \frac{\partial}{\partial x^k}"\Gamma^i_{rj} - \frac{\partial}{x^j}"\Gamma^i_{rk} + "\Gamma^s_{rj}"\Gamma^i_{sk} - "\Gamma^s_{rk}"\Gamma^i_{sj},$$

$$\overset{m}{R}{}^i_{rkj} = \frac{\partial}{\partial x^k}"\overset{m}{\Gamma}{}^i_{rj} - \frac{\partial}{x^j}"\overset{m}{\Gamma}{}^i_{rk} + "\overset{m}{\Gamma}{}^s_{rj}"\overset{m}{\Gamma}{}^i_{sk} - "\overset{m}{\Gamma}{}^s_{rk}"\overset{m}{\Gamma}{}^i_{sj}.$$

It is ẹase to see that

$$"R^i_{rkj} = \overset{m}{R}{}^i_{rkj} + \overset{m}{\nabla}_k H^i_{rj} - \overset{m}{\nabla}_j H^i_{rk} + H^s_{rj}H^i_{sk} - H^s_{rk}H^i_{sj}.$$

Taking into account (2.1) and the fact that $"\overset{m}{\Gamma}$ is a metric connection, we obtain

$$\begin{aligned}"R_{irkj} &= \overset{m}{R}_{irkj} + \frac{1}{2}(\overset{m}{\nabla}_j\gamma_k - \overset{m}{\nabla}_k\gamma_j)Q_{ir} + \frac{1}{2}\gamma_k\overset{m}{\nabla}_j Q_{ir} - \frac{1}{2}\gamma_j\overset{m}{\nabla}_k Q_{ir}\\
&- \frac{1}{2}g_{ij}\overset{m}{\nabla}_k\tilde{\gamma}_r + \frac{1}{2}g_{ik}\overset{m}{\nabla}_j\tilde{\gamma}_r + \frac{1}{2}g_{jr}\overset{m}{\nabla}_k\tilde{\gamma}_i - \frac{1}{2}g_{kr}\overset{m}{\nabla}_j\tilde{\gamma}_i\\
(2.2)\qquad &+ \frac{1}{4}(\gamma_j\tilde{\gamma}_p Q^p_r g_{ik} - \gamma_j\tilde{\gamma}_i Q_{rk} + \gamma_k\tilde{\gamma}_r Q_{ij} - \gamma_k\tilde{\gamma}^p Q_{ip}g_{ir}\\
&- \gamma_k\tilde{\gamma}_p Q^p_r g_{ij} + \gamma_k\tilde{\gamma}_i Q_{rj} - \gamma_j\tilde{\gamma}_r Q_{ik} + \gamma_j\tilde{\gamma}^p Q_{ip}g_{kr})\\
&+ \frac{1}{4}(\tilde{\gamma}_r\tilde{\gamma}_j g_{ik} - g_{jr}g_{ik}\tilde{\gamma}_p\tilde{\gamma}^p + \tilde{\gamma}_i\tilde{\gamma}_k g_{jr}\\
&- \tilde{\gamma}_r\tilde{\gamma}_k g_{ij} + g_{kr}g_{ij}\tilde{\gamma}_p\tilde{\gamma}^p - \tilde{\gamma}_i\tilde{\gamma}_j g_{kr}).\end{aligned}$$

Interchanging in (2.2) the place of the indices i and r and that for the indices k and j and adding the obtained relation to (2.2), we get

$$(2.3)\qquad {}''R_{irkj} + {}''R_{rijk} = \overset{m}{R}_{irkj} + \overset{m}{R}_{rijk} - g_{ij}\Theta_{kr} + g_{ik}\Theta_{jr} + g_{jr}\Theta_{ki} - g_{kr}\Theta_{ji} ,$$

where

$$\Theta_{kr} = \overset{m}{\nabla}_k\gamma_r + \tilde{\gamma}_r\tilde{\gamma}_k - \frac{1}{2} g_{kr}\tilde{\gamma}_p\tilde{\gamma}^p .$$

Introducing the notations

$${}''R_{rk} = g^{ij}({}''R_{irkj} + {}''R_{rijk}), \quad \overset{m}{R}_{rk} = g^{ij}(\overset{m}{R}_{irkj} + \overset{m}{R}_{rijk})$$

$${}''R = g^{rk}\ {}''R_{rk} , \qquad \overset{m}{R} = g^{rk}\overset{m}{R}_{rk} ,$$

and transvecting (2.3) with g^{ij}, we find

$$(2.4)\qquad {}''R_{rk} = \overset{m}{R}_{rk} + (2-n)\Theta_{kr} - g_{kr}\Theta_{ji}g^{ji} .$$

Transvecting (2.4) with g^{rk}, we obtain

$$\Theta_{ab}g^{ab} = \frac{1}{2(1-n)} ({}''R - \overset{m}{R}) .$$

Substituting this into (2.4), we get

$$(2.5)\qquad \Theta_{kr} = \frac{1}{2-n} ({}''R_{rk} - \overset{m}{R}_{rk}) + \frac{g_{rk}}{2(1-n)(2-n)} ({}''R - \overset{m}{R}) .$$

Taking into account (2.5), we express (2.3) as follows:

$$(2.6)\qquad \begin{aligned} &{}''R_{irkj} + {}''R_{rijk} + \frac{1}{2-n}(g_{ij}{}''R_{rk} - g_{ik}{}''R_{rj} - g_{jr}{}''R_{ik} + g_{kr}{}''R_{ij}) \\ &+ \frac{{}''R}{(n-1)(n-2)} (g_{ij}g_{kr} - g_{ik}g_{rj}) = \\ &= \overset{m}{R}_{irkj} + \overset{m}{R}_{rijk} + \frac{1}{2-n} (g_{ij}\overset{m}{R}_{rk} - g_{ik}\overset{m}{R}_{rj} - g_{jr}\overset{m}{R}_{ik} + g_{kr}\overset{m}{R}_{ij}) \\ &+ \frac{\overset{m}{R}}{(n-1)(n-2)} (g_{ij}g_{kr} - g_{ik}g_{rj}) . \end{aligned}$$

The tensor on the right-hand side of (2.6) does not depend on the vector γ_i. Thus we have

THEOREM 1. *The tensor*

$$"R_{irkj} + "R_{rijk} + \frac{1}{n-2}(g_{ik}"R_{rj} - g_{ij}"R_{rk} + g_{jr}"R_{ik} - g_{kr}"R_{ij})$$

$$+ \frac{"R}{(n-1)(n-2)}(g_{ij}g_{kr} - g_{ik}g_{rj})$$

does not depend on the vector field γ_i *,i.e. it is the same for all* $W-O_n$ *spaces of the second kind.*

3. WEYL - OTSUKI SPACES OF THE THIRD KIND

In this case $m_{ij} = P_{ia}P^a_j$, and connection (1.5) has the form:

$$(3.1) \quad "\Gamma^i_{jk} = "\overset{m}{\Gamma}{}^i_{jk} + \frac{1}{2}(\gamma_k\delta^i_j + P^i_k\tilde{\gamma}_j - P_{jk}\tilde{\gamma}^i)$$

where

$$\tilde{\gamma}_i = \gamma_a Q^a_i \quad , \quad \tilde{\gamma}^i = \tilde{\gamma}_a g^{ai} \quad ,$$

while connection (1.6) has the form

$$(3.2) \quad {}'\Gamma^i_{jk} = {}'\overset{m}{\Gamma}{}^i_{jk} - \frac{1}{2}(\gamma_k\delta^i_j + \gamma_j\delta^i_k - \gamma_q Q^q_s Q^{is} P_{ja}P^a_k) \ .$$

Let the metric tensor g_{ij} now undergoe the conformal transformation

$$\bar{g}_{ij} = e^{2\sigma}g_{ij} \ , \quad \bar{g}^{ij} = e^{-2\sigma}g^{ij} \ .$$

Then the Christoffel symbols formed with respect to the two tensors are related as follows

$$\overline{\{{}_j{}^i{}_k\}} = \{{}_j{}^i{}_k\} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - g_{jk}\sigma^i \ , \quad \sigma_i = \frac{\partial\sigma}{\partial x^i}, \sigma^i = g^{ia}\sigma_a$$

Obviously the basic tensor P and the basic vector γ are invariant under conformal transformationm because these are independent

from g_{ij}, i.e.

$$\bar{P}^i_j = P^i_j\ ,\quad \bar{Q}^i_j = Q^i_j\ ,\quad \bar{\gamma}_i = \gamma_i$$

Then $''\bar{\Gamma}^i_{jk}$ and $'\bar{\Gamma}^i_{jk}$ can be expressed in the form

$$(3.3)\qquad ''\bar{\Gamma}^i_{jk} = ''\overset{m}{\bar{\Gamma}}{}^i_{jk} + \frac{1}{2}\,(\gamma_k\delta^i_j + P^i_k\tilde{\gamma}_j - P_{jk}\tilde{\gamma}^i)\ ,$$

$$(3.4)\qquad '\bar{\Gamma}^i_{jk} = '\overset{m}{\bar{\Gamma}}{}^i_{jk} - \frac{1}{2}\,(\gamma_k\delta^i_j + \gamma_j\delta^i_k - \gamma_q Q^q_s Q^{is} P_{ja} P^a_k\,).$$

Denoting by $\overset{o}{\bar{\nabla}}$ the ordinary covariant derivative with respact to $\overline{\left\{{i \atop j\,k}\right\}}$, and taking into account that

$$\overset{o}{\bar{\nabla}}_k P^i_j = \overset{o}{\nabla}_k P^i_j + \delta^i_k\sigma_s P^s_j - P_{jk}\sigma^i - P^i_k\sigma_j + g_{jk}P^i_s\sigma^s\ ,$$

we easely find

$$''\overset{m}{\bar{\Gamma}}{}^i_{jk} = ''\overset{m}{\Gamma}{}^i_{jk} + \delta^i_j\sigma_k + P^i_k\sigma_a Q^a_j - P_{kj}\sigma_a Q^{ai}\ ,$$

$$'\overset{m}{\bar{\Gamma}}{}^i_{jk} = '\overset{m}{\Gamma}{}^i_{jk} + \delta^i_j\sigma_k + \delta^i_k\sigma_j - \sigma_q Q^q_s Q^{is} P_{ja} P^a_k\ .$$

Substituting this into (3.3) respective (3.4), we get

$$(3.5)\qquad ''\bar{\Gamma}^i_{jk} = ''\overset{m}{\Gamma}{}^i_{jk} + \delta^i_j(\sigma_k + \tfrac{1}{2}\gamma_k) + P^i_k(\sigma_a + \tfrac{1}{2}\gamma_a)Q^a_j - P_{jk}(\sigma_a + \tfrac{1}{2}\gamma_a)Q^{ai}\ ,$$

$$(3.6)\qquad '\bar{\Gamma}^i_{jk} = '\overset{m}{\Gamma}{}^i_{jk} + \delta^i_j(\sigma_k - \tfrac{1}{2}\gamma_k) + \delta^i_k(\sigma_j - \tfrac{1}{2}\gamma_j) - P_{ja}P^a_k Q^q_s Q^{is}(\sigma_q - \tfrac{1}{2}\gamma_q)\ .$$

Comparing (3.1) with (3.5), and (3.2) with (3.6), we see that:

Under conformal transformation, each of the connections $'\Gamma$, $''\Gamma$ of a **W** *-O_n -space of the third kind transforms into the connection of the same form.*

We suppose now that one of the conditions

(A) $\overset{o}{\nabla}_k P^i_j = \pi_k P^i_j$;

(B) $\overset{o}{\nabla}_k P_{ij} = \pi_i P_{kj} + \pi_j P_{ki}$ (or, equivalently, $\overset{o}{\nabla}_k P^i_j = \pi^i P_{jk} + \pi_j P^i_k$)

is satisfied.

First, we investigate connection (3.5). Taking into account (1.7), it is easy to see that

$$\overset{m}{{}''\Gamma}{}^{i}_{jk} = \{{}^{\,i}_{j\,k}\} + \varepsilon\pi_a Q^q_j P^i_k - \varepsilon\pi_a Q^{ai} P_{kj} \;,$$

where $\varepsilon = +1$ if condition (A) is satisfied, and $\varepsilon = -1$, if condition (B) is satisfied. Substituting this into (3.5), we obtain

$$''\bar{\Gamma}^i_{jk} = \{{}^{\,i}_{j\,k}\} + \delta^i_j(\sigma_k + \tfrac{1}{2}\gamma_k) + P^i_k(\sigma_a + \tfrac{1}{2}\gamma_a + \varepsilon\pi_a)Q^a_j$$

$$- P_{jk}(\sigma_a + \tfrac{1}{2}\gamma_a + \varepsilon\pi_a)Q^{ai} \;, \tag{3.7}$$

If we put

$$\sigma_k + \tfrac{1}{2}\gamma_k = S_k \;, \qquad (\sigma_a + \tfrac{1}{2}\gamma_a + \varepsilon\pi_a)Q^a_j = \tilde{S}_j \;,$$

we may express (3.7) in the form

$$''\bar{\Gamma}^i_{jk} = \{{}^{\,i}_{j\,k}\} + \delta^i_j S_k + P^i_k\tilde{S}_j - P_{kj}\tilde{S}^i \;.$$

Let us denote by $''\bar{R}^i_{rkj}$ the curvature tensor of connection $''\bar{\Gamma}$, and by K^i_{rkj} – the curvature tensor of connection $\{{}^{\,i}_{j\,k}\}$. Then we have

$$\begin{aligned} ''\bar{R}_{irkj} &= K_{irkj} + g_{ir}(\overset{o}{\nabla}_k S_j - \overset{o}{\nabla}_j S_k) \\ &+ P_{ij}(\overset{o}{\nabla}_k\tilde{S}_r + \varepsilon\tilde{S}_r\pi_k - \tilde{S}_r P^p_k S_p + \tfrac{1}{2} P_{kr}\tilde{S}^p\tilde{S}_p) \\ &- P_{ik}(\overset{o}{\nabla}_j\tilde{S}_r + \varepsilon\tilde{S}_r\pi_j - \tilde{S}_r P^p_j\tilde{S}_p + \tfrac{1}{2}P_{jr}\tilde{S}^p\tilde{S}_p) \\ &- P_{jr}(\overset{o}{\nabla}_k\tilde{S}_i + \varepsilon\tilde{S}_i\pi_k - \tilde{S}_i P^p_k\tilde{S}_p + \tfrac{1}{2}P_{ik}\tilde{S}^p\tilde{S}_p) \\ &+ P_{kr}(\overset{o}{\nabla}_j\tilde{S}_i + \varepsilon\tilde{S}_i\pi_j - \tilde{S}_i P^p_j\tilde{S}_p + \tfrac{1}{2} P_{ij}\tilde{S}^p\tilde{S}_p) \;. \end{aligned} \tag{3.8}$$

If we interchange the indices i and r and the indices k and j and add the obtained relation to (3.8), we get:

$$\tfrac{1}{2}(''\bar{R}_{arkj} + ''\bar{R}_{rajk}) = K_{arkj} + P_{rk}\Psi_{ja} - P_{rj}\Psi_{ka} - P_{ka}\Psi_{jr} + P_{ja}\Psi_{kr} \;,$$

or, transvecting with g^{ia}

$$\frac{1}{2}({}''\bar{R}_{arkj} + {}''\bar{R}_{rajk})g^{ia} =$$

(3.9)
$$= K^{i}_{rkj} + P_{rk}\Psi^{i}_{j} - P_{rj}\Psi^{i}_{k} - P^{i}_{k}\Psi_{jr} + P^{i}_{j}\Psi_{kr} ,$$

where

$$\Psi_{ji} = \overset{o}{\nabla}_j\tilde{S}_i + \varepsilon\tilde{S}_i\pi_j - \tilde{S}_i P^{p}_{j}\tilde{S}_p + \frac{1}{2} P_{ji}\tilde{S}^p\tilde{S}_p , \quad \Psi^{i}_{j} = \Psi_{ja}g^{ai} .$$

Introducing the notations

$${}''\bar{R}^{*}_{kr} = \frac{1}{2}({}''\bar{R}_{arkb} + {}''\bar{R}_{rabk})Q^{ab} , \quad {}''\bar{R}^{*r}_{k} = {}''\bar{R}^{*}_{ka}\, g^{ar} ,$$

$$\overset{*}{K}_{kr} = K^{a}_{rkb}Q^{b}_{a} , \qquad \overset{*}{K}{}^{r}_{k} = \overset{*}{K}_{ka}g^{ar} ,$$

$${}''\bar{R}^{*} = {}''\bar{R}^{*}_{kr}Q^{kr} , \qquad \overset{*}{K} = \overset{*}{K}_{kr}Q^{kr} ,$$

and transvecting (3.9) with Q^{j}_{i} , we find:

(3.10)
$${}''\bar{R}^{*}_{kr} = \overset{*}{K}_{kr} + (n-2)\Psi_{kr} + P_{kr}\Psi_{ji}Q^{ji} .$$

Transvecting (3.10) with Q^{rk} , we have

$$\Psi_{ji}Q^{ji} = \frac{1}{2(n-1)} ({}''\bar{R}^{*} - \overset{*}{K}).$$

Substituting this into (3.10) , we get

$$\Psi_{kr} = \frac{1}{n-2} ({}''\bar{R}^{*}_{kr} - \overset{*}{K}_{kr}) - \frac{P_{kr}}{2(n-1)(n-2)} ({}''\bar{R}^{*} - \overset{*}{K}).$$

Finally, inserting this into (3.9), we obtain:

(3.11)
$$\begin{aligned}&\frac{1}{2}({}''\bar{R}_{arkj} + {}''\bar{R}_{rajk})g^{ia} + \frac{1}{n-2} (P_{rj}\,{}''\bar{R}^{*i}_{k} - P_{rk}\,{}''\bar{R}^{*i}_{j} + P^{i}_{k}\,{}''\bar{R}^{*}_{jr} - \\ &- P^{i}_{j}\,{}''\bar{R}^{*}_{kr}) - \frac{{}''\bar{R}^{*}}{(n-1)(n-2)}(P_{rj}\, P^{i}_{k} - P_{rk}\, P^{i}_{j}) = \\ &= K^{i}_{rkj} + \frac{1}{n-2} (P_{rj}\overset{*}{K}{}^{i}_{k} - P_{rk}\overset{*}{K}{}^{i}_{j} + P^{i}_{k}\, \overset{*}{K}_{jr} - P^{i}_{j}\overset{*}{K}_{kr}) \\ &- \frac{\overset{*}{K}}{(n-1)(n-2)} (P_{rj}\, P^{i}_{k} - P_{rk}\, P^{i}_{j}).\end{aligned}$$

The tensor on the right-hand side of (3.11) depends only on the basic tensor P^i_j and g_{ij}. Therefore, we have

THEOREM 2. *If condition (A) or condition (B) is satisfied, the tensor on the left-hand side of (3.11) is invariant with respect to the conformal transformation. This tensor does not depend on the vectors* π_i *and* γ_i *as well.*

We are to investingate connection (3.6). Taking into account (1.8) and condition (A), we find

$$'\bar{\Gamma}^i_{jk} = \{{}_j{}^i{}_k\} + \delta^i_j(\sigma_k - \tfrac{1}{2}\gamma_k + \pi_k) + \delta^i_k(\sigma_j - \tfrac{1}{2}\gamma_j + \pi_j) - \\ - P_{ja}P^a_k Q^q_s Q^{is}(\sigma_q - \tfrac{1}{2}\gamma_q + \pi_q) .$$

Putting

$$\sigma_k - \frac{1}{2}\gamma_k + \pi_k = V_k ,$$

we may re-write this connection as follows:

$$'\bar{\Gamma}^i_{jk} = \{{}_j{}^i{}_k\} + \delta^i_j V_k + \delta^i_k V_j - P_{ja}P^a_k Q^q_s Q^{is} V_q . \tag{3.12}$$

Let us denote by $'\bar{R}^i_{rkj}$ the curvature tensor of the connection $'\bar{\Gamma}$. Then we have

$$\begin{aligned} '\bar{R}^i_{rkj} &= K^i_{rkj} + \delta^i_r(\overset{o}{\nabla}_k V_j - \overset{o}{\nabla}_j V_k) \\ &+ \delta^i_j(\overset{o}{\nabla}_k V_r - V_r V_k + \tfrac{1}{2}V_l V_p Q^l_s Q^{ps} P_{ra} P^a_k) \\ &- \delta^i_k(\overset{o}{\nabla}_j V_r - V_r V_j + \tfrac{1}{2}V_l V_p Q^l_s Q^{ps} P_{ra} P^a_j) \\ &+ P_{ra}P^a_k(\overset{o}{\nabla}_j V_l - V_j V_l + \tfrac{1}{2}V_m V_p Q^m_s Q^{ps} P_{la} P^a_j) Q^{it} Q^l_t \\ &- P_{ra}P^a_j(\overset{o}{\nabla}_k V_l - V_k V_l + \tfrac{1}{2}V_m V_p Q^m_s Q^{ps} P_{la} P^a_k) Q^{it} Q^l_t \end{aligned} \tag{3.13}$$

because of

$$\overset{o}{\nabla}_k(Q^l_s Q^i_t P^a_r P^b_j) = \overset{o}{\nabla}_j(Q^l_s Q^i_t P^a_r P^b_k) = 0$$

and

$$P_{la}P^a_j Q^l_s Q^{is} = \delta^i_j , \quad P_{la}P^a_k Q^{is} Q^l_s = \delta^i_k .$$

Contracting with respect to i and r, we get

$$\overset{o}{\nabla}_k V_j - \overset{o}{\nabla}_j V_k = \frac{1}{n}\ {}'\bar{R}^a{}_{akj} .$$

Substituting this into (3.13) and putting

$$\phi_{kr} = \overset{o}{\nabla}_k V_r - V_k V_r + \frac{1}{2} V_l V_p Q^l_s Q^{ps} P_{ra} P^a_k ,$$

we have

$$\begin{aligned} {}'\bar{R}^i{}_{rkj} - \frac{1}{n}\ \delta^i_r\, {}'\bar{R}^r{}_{akj} &= K^i_{rkj} + \delta^i_j\, \phi_{kr} - \delta^i_k \phi_{jr} \\ &+ P_{ra}P^a_k \phi_{jl} Q^{is} Q^l_s - P_{ra}P^a_j \phi_{kl} Q^{is} Q^l_s . \end{aligned} \tag{3.14}$$

Introducting the notation

$${}'\bar{R}_{rk} = {}'\bar{R}^a{}_{rka} - \frac{1}{n}\ {}'\bar{R}^a{}_{akr} ,$$

and contracting (3.14) for i and j, we obtain

$${}'\bar{R}_{rk} = K_{rk} + (n-2)\phi_{kr} + P_{rt}P^t_k Q^b_a Q^p_b \phi^a_p .$$

Transvecting this with $Q^{pr}Q^k_p$, we find

$$Q^{ba}Q^p_b \phi_{pa} = \frac{1}{2(n-1)}\ ({}'\bar{R}_{rk}Q^{pr}Q^k_p - K_{rk}Q^{pr}Q^k_p) .$$

Thus we have

$$\phi_{kr} = \frac{1}{n-2}\ ({}'\bar{R}_{rk} - K_{rk}) - \frac{P_{rt}P^t_k}{2(n-1)(n-2)}\ ({}'\bar{R}_{ab}Q^{pa}Q^b_p - K_{ab}Q^{pa}Q^b_p) .$$

Substituting this into (3.14), we obtain

$$
(3.15)\qquad
\begin{aligned}
&{}'\bar{R}^{i}_{rkj} - \frac{1}{n}\,\delta^{i}_{r}\,{}'\bar{R}^{a}_{akj} - \frac{1}{n-2}\,(\delta^{i}_{j}\,{}'\bar{R}_{rk} - \delta^{i}_{k}\,{}'\bar{R}_{rj} - P_{rt}P^{t}_{j}Q^{ba}Q^{i}_{b}\,{}'\bar{R}_{ak} \\
&\qquad + P_{tr}P^{t}_{k}Q^{ba}Q^{i}_{b}\,{}'\bar{R}_{aj}) \\
&\quad + \frac{{}'\bar{R}_{ab}Q^{pa}Q^{b}_{p}}{(n-1)(n-2)}\,(\delta^{i}_{j}P_{rt}P^{t}_{k} - \delta^{i}_{k}P_{rt}P^{t}_{j}) = \\
&\quad = K^{i}_{rkj} - \frac{1}{n-2}\,(\delta^{i}_{j}K_{rk} - \delta^{i}_{k}K_{rj} - P_{rt}P^{t}_{j}Q^{ba}Q^{i}_{b}K_{ak} \\
&\qquad + P_{rt}P^{t}_{k}Q^{ba}Q^{i}_{b}K_{aj}) + \\
&\quad + \frac{K_{ab}\;Q^{pa}Q^{b}_{a}}{(n-1)(n-2)}\,(\delta^{i}_{j}P_{rt}P^{t}_{k} - \delta^{i}_{k}P_{rt}P^{t}_{j})\;.
\end{aligned}
$$

In case condition (B) is satisfied, we start with (3.6), which, putting

$$\sigma_{k} - \frac{1}{2}\,\gamma_{k} = T_{k}\;,$$

can be re-written in the form

$$(3.16)\qquad {}'\bar{\Gamma}^{i}_{jk} = {}'\overset{m}{\Gamma}{}^{i}_{jk} + T_{k}\delta^{i}_{j} + T_{j}\delta^{i}_{k} - T_{a}Q^{a}_{s}Q^{is}P_{jb}P^{b}_{k}\;.$$

This form is the same as (3.12). Only, in (3.12) the connection $\left\{ {i \atop j\;k} \right\}$ depends of the basic tensor g_{ij}, while here connection ${}'\overset{m}{\Gamma}{}^{i}_{jk}$ depends of the bacic tensor g_{ij} and P^{i}_{j} (see (1.8)). Besides, denoting by $\overset{m}{\nabla}$ the ordinary covariant derivative with respect tp ${}'\overset{m}{\Gamma}{}^{i}_{jk}$, and using condition (B), we find:

$$
(3.17)\qquad
\begin{aligned}
&{}'\overset{m}{\Gamma}{}^{i}_{jk} = \left\{ {i \atop j\;k} \right\} + \pi^{a}Q^{i}_{a}P_{kj} + \pi_{a}Q^{at}Q^{i}_{t}P^{b}_{k}P_{jb}\;; \\
&\overset{o}{\nabla}_{k}Q^{i}_{j} = -\,\pi^{a}Q^{i}_{a}g_{kj} - \pi_{a}Q^{a}_{j}\delta^{i}_{k}\;; \\
&(\overset{m}{\nabla}_{k}Q^{l}_{s})Q^{is} = -\pi_{a}Q^{a}_{s}Q^{is}\delta^{l}_{k} - \pi_{a}Q^{at}Q^{h}_{t}Q^{l}_{h}P^{i}_{k}\;, \\
&Q^{l}_{s}(\overset{m}{\nabla}_{k}Q^{is}) = \pi_{a}Q^{a}_{s}Q^{is}\delta^{l}_{k} + \pi_{a}Q^{at}Q^{s}_{r}Q^{l}_{s}P^{i}_{k}\;,
\end{aligned}
$$

i.e.
$$\overset{m}{\nabla}_k(\Omega^l_s\Omega^{is}) = 0;$$

similary
$$\overset{m}{\nabla}_k(P_{ra}P^a_j) = 0 .$$

Thus, proceeding with connection (3.16) in the same manner as with (3.12), we find instead of (3.15), the relation

$$\begin{aligned}
{}'\bar{R}^i_{rkj} &- \frac{1}{n}\,\delta^i_r\,{}'\bar{R}^a_{akj} - \frac{1}{n-2}\,(\delta^i_j\,{}'\bar{R}_{rk} - \delta^i_k\,{}'\bar{R}_{rj} - P_{rt}P^t_jQ^{ba}Q^i_b\,{}'\bar{R}_{ak} \\
&\qquad + P_{rt}P^t_kQ^{ba}Q^i_b\,{}'\bar{R}_{aj}) \\
&+ \frac{{}'\bar{R}_{ab}Q^{pa}Q^b_p}{(n-1)(n-2)}\,(\delta^i_jP_{rt}P^t_k - \delta^i_kP_{rt}P^t_j) = \\
&= \overset{m}{K}{}^i_{rkj} - \frac{1}{n}\,\delta^i_r\overset{m}{K}{}^a_{akj} - \frac{1}{n-2}(\delta^i_j\,\overset{m}{K}_{rk} - \delta^i_k\,\overset{m}{K}_{rj} - P_{rt}P^t_jQ^{ba}Q^i_b\,\overset{m}{K}_{ak} \\
&\qquad + P_{rt}P^t_kQ^{ba}Q^i_b\,\overset{m}{K}_{aj}) \\
&+ \frac{\overset{m}{K}_{ab}Q^{pa}Q^b_p}{(n-1)(n-2)}\,(\delta^i_jP_{rt}P^t_k - \delta^i_kP_{rt}P^t_j) ,
\end{aligned}$$

where $\overset{m}{K}{}^i_{rkj}$ is the curvature tensor with respect to connection (3.17) and $\overset{m}{K}_{rk} = \overset{m}{K}{}^a_{rka}$.

Therefore, we have

THEOREM 3. *If condition (A) or condition (B) is satisfied, besides the tensor on the left-hand side of (3.11), the tensor*

$$\begin{aligned}
&{}'\bar{R}^i_{rkj} - \frac{1}{n}\,\delta^i_r\;{}'\bar{R}^a_{akj} \\
&- \frac{1}{n-2}(\delta^i_j\,{}'\bar{R}_{rk} - \delta^i_k\,{}'\bar{R}_{rj} - P_{rt}P^t_jQ^{ba}Q^i_b\,{}'\bar{R}_{ak} + P_{rt}P^t_kQ^{ba}Q^i_b\;{}'\bar{R}_{aj}) \\
&\qquad + \frac{{}'\bar{R}_{ab}Q^{pa}Q^b_p}{(n-1)(n-2)}\,(\delta^i_jP_{rt}P^t_k - \delta^i_kP_{rt}P^t_j)
\end{aligned}$$

is invariant with respect to the conformal transformation, too. This tensor does not depend on the vector field γ_i *and if condition (A) is satisfied it does not depend on the vector field* π_i *either.*

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REZIME

WEYL-OTSUKI-JEVI PROSTORI DRUGE I TREĆE VRSTE

U ovom radu ispituje se ona opšta regularna koneksija Otsuki-jevog prostora koja zadovoljava uslove a´) i c) i čiji je kontravarijantni deo $'\Gamma$ simetričan. Ta je koneksija oblika (1.5),(1.6),(1.7) i (1.8). Ako je, pri tom, $m_{ij}=P_{ij}$, odnosno $m_{ij}=P_p P^a_j$, posmatrani prostor je Weyl-Otsuki-jev prostor (W-O_n-prostor) druge odnosno treće vrste.

U §2 je dokazano da je tenzor (2.6) zajednički za sve W-O_n -prostore druge vrste.

U §3 ispituju se konformne transformacije W-O_n -prostora treće vrste. Dokažana je teorema:

Ako je zadovoljen uslov (A) ili uslov (B), tenzor (3.11) je invarijantan u odnosu na konformne transformacije. Taj tenzor ne zavisi ni od polja vektora π_i i γ_i .

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CONNECTIONS BETWEEN THE DOUBLE ALTERNATED ABSOLUTE DIFFERENTIAL OF CURVATURE TENSORS OF THE FINSLER SPACE AND INDUCED CURVATURE TENSORS OF ITS SUBSPACE

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1. INTRODUCTION

The subspace F_m of a Finsler space F_n is given by the equations:

$$x^i = x^i(u^1, u^2, \dots, u^m) \qquad i,j,k,l,\dots = 1,2,\dots,n,$$

if the rank of the matrix

$$\| B^i_\alpha \| = \| \frac{\partial x_i}{\partial u^\alpha} \| \qquad \alpha,\beta,\gamma,\delta,\varepsilon,\iota, \ ,\eta,\dots,=1,2,\dots,m$$

is assumed to be m. To every vector $\dot u^\alpha$, which is tangent to F_m, may be associated a vector $\dot x^i$ in the following way:

$$\dot x^i = B^i_\alpha \dot u^\alpha \ .$$

At every point P of F_m there are n-m linearly independent vectors

$$\overset{\nu}{N}_i \qquad \mu,\nu,\zeta,\rho,\Psi = n-m+1,\dots,n$$

which satisfy the relations of |1|

$$\overset{\nu}{N}_i B^i_\alpha = 0 \qquad \underset{\mu}{N}^i \overset{\text{def}}{=\!=} g^{ij}(x,\dot x)\overset{\mu}{N}_j$$

$$g_{ij}(x,\dot x)\underset{\nu}{N}^i \underset{\mu}{N}^j = \delta_{\nu\mu}$$

If D and Δ are absolute differentials which correspond to the motion from $(u^\beta, \dot u^\beta)$ to $(u^\beta + du^\beta, \dot u + d\dot u^\beta)$ and $(u^\beta + \delta u^\beta, \dot u^\beta + \delta \dot u^\beta)$

in the subspace F_m of a Finsler space F_n then as in |2|* we have

(1.1) $[\Delta D] B^i_\alpha = \bar{\Omega}_\alpha{}^\delta (d,\delta) B^i_\delta + \bar{\Omega}_\alpha{}^\mu (d,\delta) \underset{\mu}{N}^i + \tilde{D} B^i_\alpha$

(1.2) $[\Delta D] \underset{\mu}{N}^i = \bar{\Omega}_\mu{}^\delta (d,\delta) B^i_\delta + \bar{\Omega}_\mu{}^\nu (d,\delta) \underset{\nu}{N}^i + \tilde{D} \underset{\mu}{N}^i$

(1.3) $\bar{\Omega}_\alpha{}^\delta (d,\delta) = \frac{1}{2} \overset{2}{\bar{R}}_\alpha{}^\delta{}_{\beta\gamma} [du^\beta \delta u^\gamma] + \overset{2}{\bar{P}}_\alpha{}^\delta{}_{\beta\gamma} [du^\beta \bar{\Delta} \ell^\gamma] + \frac{1}{2} \overset{2}{\bar{S}}_\alpha{}^\delta{}_{\beta\gamma} [\bar{D}\ell^\beta \bar{\Delta}\ell^\gamma]$

(1.4) $\bar{\Omega}_\alpha{}^\mu (d,\delta) = \frac{1}{2} \overset{2}{\bar{R}}_\alpha{}^\mu{}_{\beta\gamma} [du^\beta \delta u^\gamma] + \overset{2}{\bar{P}}_\alpha{}^\mu{}_{\beta\gamma} [du^\beta \bar{\Delta} \ell^\gamma] + \frac{1}{2} \overset{2}{\bar{S}}_\alpha{}^\mu{}_{\beta\gamma} [\bar{D}\ell^\beta \bar{\Delta}\ell^\gamma]$

(1.5) $\bar{\Omega}_{\mu\alpha} = - \bar{\Omega}_{\alpha\mu}$

(1.6) $\bar{\Omega}_\mu{}^\nu (d,\delta) = \frac{1}{2} \overset{2}{\bar{R}}_\mu{}^\nu{}_{\beta\gamma} [du^\beta \delta u^\gamma] + \overset{2}{\bar{P}}_\mu{}^\nu{}_{\beta\gamma} [du^\beta \bar{\Delta} \ell^\gamma] + \frac{1}{2} \overset{2}{\bar{S}}_\mu{}^\nu{}_{\beta\gamma} [\bar{D}\ell^\beta \bar{\Delta}\ell^\gamma]$

For the arbitrary vector field ξ defined on the subspace F_m of the Funsler space F_n we have

(1.7) $\xi^i = B^i_\alpha \xi^\alpha + \underset{\mu}{N}^i \xi^\mu$

It is known that

(1.8) $[\Delta D] \xi^i = \frac{1}{2} R_j{}^i{}_{hk} \xi^j [dx^h \delta x^k] + P_j{}^i{}_{hk} \xi^j [dx^h \Delta \ell^k] +$
$+ \frac{1}{2} S_j{}^i{}_{hk} \xi^j [D\ell^h \Delta \ell^k] + \tilde{D} \xi^i$

where

(1.9) $dx^h = B^h_\alpha du^\alpha$

(1.10) $D\ell^k = B^k_\beta \bar{D}\ell^\beta + \bar{H}^k_\beta du^\beta$

On the other hand

(1.11) $[\Delta D] \xi^i = \xi^\alpha [\Delta D] B^i_\alpha + \xi^\mu [\Delta D] \underset{\mu}{N}^i + (\delta d - d\delta) \xi^\alpha B^i_\alpha \; (\delta d - d\delta) \xi^\mu \underset{\mu}{N}^i$

*) One can easely conclude what the tensors $\overset{2}{R}, \overset{2}{P}, \overset{2}{s}$ are from (2.1), (29´), (2.18) and (2.19), (2,27), (2.33) in |2| . They are explicitly defined in |3| .

Substituting (1.9), (1.10) into (1.8) and (1.1)-(1.6) into (1.11) we get two relations for $[\Delta D]\xi^{i}$. Equating the coefficients of bivectors

$[du^{\beta}\ \delta u^{\gamma}], [du^{\beta}\bar{\Delta}\ell^{\gamma}]$ and $[\bar{D}\ell^{\beta}\bar{\Delta}\ell^{\gamma}]$, neglecting inifinitesimals of a higher order, we obtain the relations:

$$(1.12)\quad R_{j\ hk}^{\ i}\xi^{j}B^{hk}_{\beta\gamma} + P_{j}^{\ i}{}_{hk}\xi^{j}B^{h}_{[\beta}\ \bar{H}^{k}_{\gamma]} + S_{j\ hk}^{\ i}\xi^{j}\bar{H}^{k}_{\beta}\ \bar{H}^{k}_{\gamma} =$$

$$= \overset{2}{R}_{\alpha\ \beta\gamma}^{\ \varepsilon}\xi^{\alpha}B^{i}_{\varepsilon} + \overset{2}{R}_{\alpha\ \beta\gamma}^{\ \mu}\xi^{\alpha}\underset{\mu}{N}^{i} + \overset{2}{R}_{\mu\ \beta\gamma}^{\ \varepsilon}\xi^{\mu}B^{i}_{\varepsilon} + \overset{2}{R}_{\mu\ \beta\gamma}^{\ \nu}\xi^{\mu}\underset{\nu}{N}^{i},$$

$$(1.13)\quad P_{j\ hk}^{\ i}\xi^{j}B^{hk}_{\beta\gamma} + S_{j\ hk}^{\ i}\xi^{j}\ \bar{H}^{h}_{\beta}B^{k}_{\gamma} =$$

$$\overset{2}{P}_{\alpha\ \beta\gamma}^{\ \varepsilon}\xi^{\alpha}B^{i}_{\varepsilon} + \overset{2}{P}_{\alpha\ \beta\gamma}^{\ \mu}\xi^{\alpha}\underset{\mu}{N}^{i} + \overset{2}{P}_{\mu\ \beta\gamma}^{\ \varepsilon}\xi^{\mu}B^{i}_{\varepsilon} + \overset{2}{P}_{\mu\ \beta\gamma}^{\ \nu}\xi^{\mu}\ \underset{\nu}{N}^{i},$$

$$(1.14.)\quad S_{j\ kh}^{\ i}\ \xi^{j}\ B^{hk}_{\beta\gamma} =$$

$$\overset{2}{S}_{\alpha\ \beta\gamma}^{\ \varepsilon}\xi^{\alpha}B^{i}_{\varepsilon} + \overset{2}{S}_{\alpha\ \beta\gamma}^{\ \mu}\xi^{\alpha}\underset{\mu}{N}^{i} + \overset{2}{S}_{\mu\ \beta\gamma}^{\ \varepsilon}\xi^{\mu}B^{i}_{\varepsilon} + \overset{2}{S}_{\mu\ \beta\gamma}^{\ \nu}\xi^{\mu}\underset{\mu}{N}^{i}.$$

Using (1,7) and putting $\xi^{\mu}=0$, then $\xi^{\alpha}=0$ in (1.12), (1.13) and (1.14) we get

$$(1.15)\quad R_{j\ hk}^{\ i}B^{jhk}_{\alpha\beta\gamma} + P_{j\ hk}^{\ i}B^{j}_{\alpha}B^{h}_{[\beta}\bar{H}^{k}_{\gamma]} + S_{j\ hk}^{\ i}B^{j}_{\alpha}\bar{H}^{h}_{\beta}\bar{H}^{k}_{\gamma} =$$

$$= \overset{2}{R}_{\alpha\ \beta\gamma}^{\ \varepsilon}B^{i}_{\varepsilon} + \overset{2}{R}_{\alpha\ \beta\gamma}^{\ \mu}\underset{\mu}{N}^{i}$$

$$(1.16)\quad R_{j\ hk}^{\ i}N^{j}B^{hk}_{\beta\gamma} + P_{j\ hk}^{\ i}\underset{\mu}{N}^{j}B^{k}_{[\beta}\ \bar{H}^{k}_{\gamma]} + S_{j\ hk}^{\ i}\underset{\mu}{N}^{j}\bar{H}^{j}_{\beta}\ \bar{H}^{k}_{\gamma} =$$

$$= \overset{2}{R}_{\mu\ \beta\gamma}^{\ \varepsilon}B^{i}_{\varepsilon} + \overset{2}{R}_{\mu\ \beta\gamma}^{\ \nu}\underset{\nu}{N}^{i}$$

$$(1.17)\quad P_{j\ hk}^{\ i}\ B^{jhk}_{\alpha\beta\gamma} + S_{j\ hk}^{\ i}\ B^{j}_{\alpha}\ \bar{H}^{h}_{\beta}\ B^{k}_{\gamma} = \overset{2}{P}_{\alpha\ \beta\gamma}^{\ \varepsilon}B^{i}_{\varepsilon} + \overset{2}{P}_{\alpha\ \beta\gamma}^{\ \mu}\underset{\mu}{N}^{i}$$

$$(1.18)\quad P_{j\ hk}^{\ i}\underset{\mu}{N}^{j}B^{hk}_{\beta\gamma} + S_{j\ hk}^{\ i}\underset{\mu}{N}^{j}\bar{H}^{h}_{\beta}B^{k}_{\gamma} = \overset{2}{P}_{\mu\ \beta\gamma}^{\ \varepsilon}\ B^{i}_{\varepsilon} + \overset{2}{P}_{\mu\ \beta\gamma}^{\ \nu}\underset{\nu}{N}^{i}$$

$$(1.19)\quad S_{j\ hk}^{\ i}\ B^{jhk}_{\alpha\beta\gamma} = \overset{2}{S}_{\alpha\ \beta\gamma}^{\ \varepsilon}B^{i}_{\varepsilon} + \overset{2}{S}_{\alpha\ \beta\gamma}^{\ \mu}\underset{\mu}{N}^{i}$$

$$(1.20)\qquad S_{j\ hk}^{\ i}N^j_{\mu}B^{hk}_{\beta\gamma} = \overset{2}{S}{}_{\mu\ \beta\gamma}^{\ \varepsilon}B^i_{\varepsilon} + \overset{2}{S}{}_{\mu\ \beta\gamma}^{\ \nu}N^i_{\nu}$$

The curvature tensors R, P, S of the Finsler space F_n and induced curvature tensors $\overset{2}{R}$, $\overset{2}{P}$, $\overset{2}{S}$ of its subspace F_m are connected by the relations (1.15)-(1.20).

2. DOUBLE ALTERNATED ABSOLUTE DIFFERENTIALS OF CURVATURE TENSORS IN F_n

If D_1 and D_2 are the absolute differentials which correspond to the motion from $(u^\beta,\dot u^\beta)$ to $(u^\beta d_1 u^\beta, u^\beta + d_1 u^\beta)$ and $(u^\beta + d_2 u^\beta, \dot u^\beta + d_2\dot u^\beta)$ in the subspace F_m of the Finsler space F_n, then we have from (1.15)-(1.20), using (1.1), (1.2) and

$$(2.1)\qquad \bar H^k_\gamma = \bar\Theta_{\iota\ \gamma}^{\ \mu}\ell^\iota N^k_\mu ,$$

$$\begin{aligned}(2.2)\qquad & [D_1D_2]R_{j\ hk}^{\ i}B^{jhk}_{\alpha\beta\gamma} + R_{j\ hk}^{\ i}(\bar\Omega_\alpha^{\ \delta}B^j_\delta + \bar\Omega_\alpha^{\ \nu}N^j_\nu)B^{hk}_{\beta\gamma} + \\ &+ R_{j\ hk}^{\ i}B^j_\alpha(\bar\Omega_\beta^{\ \delta}B^h_\delta + \bar\Omega_\beta^{\ \nu}N^h_\nu)B^k_\gamma + R_{j\ hk}^{\ i}B^{jh}_{\alpha\beta}(\bar\Omega_\gamma^{\ \delta}B^k_\delta + \bar\Omega_\gamma^{\ \nu}N^k_\nu) + \\ &+ [D_1D_2]P_{j\ hk}^{\ i}B^{jh}_{\alpha[\beta}\bar H^k_{\gamma]} + P_{j\ hk}^{\ i}(\bar\Omega_\alpha^{\ \delta}B^j_\delta + \Omega_\alpha^{\ \nu}N^j_\nu)B^h_{[\beta}\bar H^k_{\gamma]} + \\ &+ P_{j\ hk}^{\ i}B^j_\alpha(\bar\Omega_{[\beta}^{\ \delta}B^k_{|\delta|} + \bar\Omega_{[\beta}^{\ \nu}N^j_\nu)\bar H^k_{\gamma]} + P_{j\ hk}^{\ i}B^j_\alpha B^h_{[\beta}\bar\Theta_{\iota\gamma}^{\ \mu}\ell^\iota(\bar\Omega_\mu^{\ \delta}B^k_\delta + \\ &+ \bar\Omega_\mu^{\ \nu}N^k_\nu) + [D_1D_2]S_{j\ hk}^{\ i}B^j_\alpha\bar H^h_\beta\bar H^k_\gamma + S^i_{jhk}(\bar\Omega_\alpha^{\ \delta}B^j_\delta + \\ &+ \bar\Omega_\alpha^{\ \nu}N^j_\nu)\bar H^h_\beta\bar H^k_\gamma + S_{j\ hk}^{\ i}B^j_\beta\bar\Theta_{\iota\ \beta}^{\ \mu}\ell^\iota(\bar\Omega_\mu^{\ \delta}B^h_\delta + \bar\Omega_\mu^{\ \nu}N^h_\nu)\bar H^k_\gamma + \\ &+ S_{j\ hk}^{\ i}B^j_\alpha\bar H^h_\beta\,\Theta_{\iota\ \gamma}^{\ \nu}\ell^\iota(\bar\Omega_\nu^{\ \delta}B^k_\delta + \bar\Omega_\nu^{\ \mu}N^k_\mu) = \\ &= \overset{2}{R}{}_{\alpha.\ \beta\gamma}^{\ \varepsilon}(\bar\Omega_\varepsilon^{\ \delta}B^i_\delta + \bar\Omega_\varepsilon^{\ \nu}N^i_\nu) + \overset{2}{R}{}_{\alpha\ \beta\gamma}^{\ \mu}(\bar\Omega_\mu^{\ \delta}B^i_\delta + \bar\Omega_\mu^{\ \nu}N^i_\nu),\end{aligned}$$

$$\begin{aligned}(2.3)\qquad & [D_1D_2]R_{j\ hk}^{\ i}N^j_\mu B^{hk}_{\beta\gamma} + R_{j\ hk}^{\ i}(\bar\Omega_\mu^{\ \delta}B^j_\delta + \bar\Omega_\mu^{\ \nu}N^j_\nu)B^{hk}_{\beta\gamma} + \\ &+ R_{j\ hk}^{\ i}N^j_\mu(\bar\Omega_\beta^{\ \delta}B^h_\delta + \bar\Omega_\beta^{\ \nu}N^h_\nu)B^k_\gamma + R_{j\ hk}^{\ i}N^j_\mu B^h_\beta(\bar\Omega_\gamma^{\ \delta}B^k_\delta + \bar\Omega_\gamma^{\ \nu}N^k_\nu) +\end{aligned}$$

$$+ [D_1D_2]P_j{}^i{}_{hk}N^j_\mu B^k_{[\beta}\bar{H}^k_{\gamma]} + P_j{}^i{}_{hk}(\bar{\Omega}_\mu{}^\delta B^j_\delta + \bar{\Omega}_\mu{}^\nu N^j_\nu)B^h_{[\beta}\bar{H}^k_{\gamma]} +$$

$$+ P_j{}^i{}_{hk}N^j_\mu(\bar{\Omega}_{[\beta}{}^\delta B^k_{|\delta|} + \bar{\Omega}_{[\beta}{}^\nu N^k_\nu)\bar{H}^k_{\gamma]} + P_j{}^i{}_{hk}N^j_\mu B^h_\beta\bar{\Theta}_\iota{}^\mu{}_\gamma\ell^\iota(\bar{\Omega}_\mu{}^\delta B^k_\delta +$$

$$+ \bar{\Omega}_\mu{}^\nu N^k_\nu) + [D_1D_2]S_j{}^i{}_{hk}N^j_\mu\bar{H}^h_\beta\bar{H}^k_\gamma + S_j{}^i{}_{hk}(\bar{\Omega}_\mu{}^\delta B^j_\delta + \bar{\Omega}_\mu{}^\nu N^j_\nu)\bar{H}^h_\beta\bar{H}^k_\gamma +$$

$$+ S_j{}^i{}_{hk}N^j_\mu\Theta_\iota{}^\psi{}_\beta\ell^\iota(\bar{\Omega}_\psi{}^\delta B^k_\delta + \bar{\Omega}_\psi{}^\nu N^k_\nu)\bar{H}^k_\gamma + S_j{}^i{}_{hk}N^j_\mu\bar{H}^k_\beta\,\bar{\Theta}_\iota{}^\psi{}_\gamma\ell^\iota(\bar{\Omega}_\psi{}^\delta B^k_\delta +$$

$$+ \bar{\Omega}_\psi{}^\nu N^k_\nu) = \overset{2}{R}_\mu{}^\varepsilon{}_{\beta\gamma}(\bar{\Omega}_\varepsilon{}^\delta B^i_\delta + \bar{\Omega}_\varepsilon{}^\nu N^i_\nu) + \overset{2}{R}_\mu{}^\nu{}_{\beta\gamma}(\bar{\Omega}_\nu{}^\delta B^i_\delta + \bar{\Omega}_\nu{}^\psi N^i_\psi)$$

(2.4)
$$[D_1D_2]P_j{}^i{}_{hk}B^{jhk}_{\alpha\beta\gamma} + P_j{}^i{}_{hk}(\bar{\Omega}_\alpha{}^\delta B^j_\delta + \bar{\Omega}_\alpha{}^\nu N^j_\nu)B^{hk}_{\beta\gamma} +$$

$$+ P_j{}^i{}_{hk}B^j_\alpha(\bar{\Omega}_\beta{}^\delta B^h_\delta + \bar{\Omega}_\beta{}^\nu N^h_\nu)B^k_\gamma + P_j{}^i{}_{hk}B^{jh}_{\alpha\beta}(\bar{\Omega}_\gamma{}^\delta B^k_\delta + \bar{\Omega}_\gamma{}^\nu N^k_\nu) +$$

$$+ [D_1D_2]S_j{}^i{}_{hk}B^j_\alpha\bar{H}^k_\beta B^k_\gamma + S_j{}^i{}_{hk}(\bar{\Omega}_\alpha{}^\delta B^j_\delta + \bar{\Omega}_\alpha{}^\nu N^j_\nu)\bar{H}^h_\beta B^k_\gamma +$$

$$+ S_j{}^i{}_{hk}B^j_\alpha\bar{\Theta}_\iota{}^\psi{}_\beta\ell^\iota(\bar{\Omega}_\psi{}^\delta B^h_\delta + \bar{\Omega}_\psi{}^\nu N^h_\nu)B^k_\gamma + S_{jhk}^{\;i}B^j_\alpha\bar{H}^h_\beta(\bar{\Omega}_\gamma{}^\delta B^k_\delta + \bar{\Omega}_\gamma{}^\nu N^k_\nu)$$

$$= \overset{2}{P}_\alpha{}^\varepsilon{}_{\beta\gamma}(\bar{\Omega}_\varepsilon{}^\delta B^i_\delta + \bar{\Omega}_\varepsilon{}^\nu N^i_\nu) + \overset{2}{P}_\alpha{}^\mu{}_{\beta\gamma}(\bar{\Omega}_\mu{}^\delta B^i_\delta + \bar{\Omega}_\mu{}^\nu N^i_\nu)$$

(2.5)
$$[D_1D_2]P_j{}^i{}_{hk}N^j_\mu B^{hk}_{\beta\gamma} + P_j{}^i{}_{hk}(\bar{\Omega}_\mu{}^\delta B^j_\delta + \bar{\Omega}_\mu{}^\nu N^j_\nu)B^{hk}_{\beta\gamma} +$$

$$+ P_j{}^i{}_{hk}N^j_\mu(\bar{\Omega}_\beta{}^\delta B^h_\delta + \bar{\Omega}_\beta{}^\nu N^h_\nu)B^k_\gamma + P_j{}^i{}_{hk}N^j_\mu B^h_\beta(\bar{\Omega}_\gamma{}^\delta B^k_\delta + \bar{\Omega}_\gamma{}^\nu N^k_\nu) +$$

$$+ [D_1D_2]S_j{}^i{}_{hk}N^j_\mu\bar{H}^h_\beta B^k_\gamma + S_j{}^i{}_{hk}(\bar{\Omega}_\mu{}^\delta B^j_\delta + \bar{\Omega}_\mu{}^\nu N^j_\nu)\bar{H}^h_\beta B^k_\gamma +$$

$$+ S_j{}^i{}_{hk}N^j_\mu\,\bar{\Theta}_\iota{}^\psi{}_\beta\ell^\iota(\bar{\Omega}_\psi{}^\delta B^h_\delta + \bar{\Omega}_\psi{}^\nu N^h_\nu)B^k_\gamma + S_j{}^i{}_{hk}N^j_\mu\bar{H}^h_\beta(\bar{\Omega}_\gamma{}^\delta B^k_\delta +$$

$$+ \bar{\Omega}_\gamma{}^\nu N^k_\nu) = \overset{2}{P}_\mu{}^\varepsilon{}_{\beta\gamma}(\bar{\Omega}_\varepsilon{}^\delta B^i_\delta + \bar{\Omega}_\varepsilon{}^\nu N^i_\nu) + \overset{2}{P}_\mu{}^\psi{}_{\beta\gamma}(\bar{\Omega}_\psi{}^\delta B^i_\delta + \bar{\Omega}_\psi{}^\nu N^i_\nu)$$

(2.6)
$$[D_1D_2]S_j{}^i{}_{hk}B^{jhk}_{\alpha\beta\gamma} + S_j{}^i{}_{hk}(\bar{\Omega}_\alpha{}^\delta B^j_\delta + \bar{\Omega}_\alpha{}^\nu N^j_\nu)B^{hk}_{\beta\gamma} + S_j{}^i{}_{hk}B^j_\alpha(\bar{\Omega}_\beta{}^\delta B^h_\delta +$$

$$+ \bar{\Omega}_\alpha{}^\nu N^h_\nu)B^k_\gamma + S_j{}^i{}_{hk}B^{jh}_{\alpha\beta}(\bar{\Omega}_\gamma{}^\delta B^k_\delta + \bar{\Omega}_\gamma{}^\nu N^k_\nu) = \overset{2}{S}_\alpha{}^\varepsilon{}_{\beta\gamma}(\bar{\Omega}_\varepsilon{}^\delta B^i_\delta + \bar{\Omega}_\varepsilon{}^\nu N^i_\nu) +$$

$$+ \overset{2}{S}_\alpha{}^\mu{}_{\beta\gamma}(\bar{\Omega}_\mu{}^\delta B^i_\delta + \bar{\Omega}_\mu{}^\nu N^i_\nu)$$

$$(2.7)\quad [D_1\ D_2]S_{j\ hk}^{\ i}\ \underset{\mu}{N}^j\ B^{hk}_{\beta\gamma} + S_{j\ hk}^{\ i}(\bar{\Omega}_{\mu}^{\ \delta}B^j_{\delta} + \bar{\Omega}_{\mu}^{\ \nu}\underset{\nu}{N}^j)B^{hk}_{\beta\gamma} + $$
$$+ S_{j\ hk}^{\ i}\underset{\mu}{N}^j(\bar{\Omega}_{\beta}^{\ \delta}B^h_{\delta} + \bar{\Omega}_{\beta}^{\ \nu}\underset{\nu}{N}^h)B^k_{\gamma} + S_{j\ hk}^{\ i}\underset{\mu}{N}^j B^h_{\beta}(\bar{\Omega}_{\gamma}^{\ \delta}B^k_{\delta} + \Omega_{\gamma}^{\ \nu}\underset{\nu}{N}^k) =$$
$$= \overset{2}{S}_{\mu\ \beta\gamma}^{\ \varepsilon}(\bar{\Omega}_{\varepsilon}^{\ \delta}B^i_{\delta} + \bar{\Omega}_{\varepsilon}^{\ \nu}\underset{\nu}{N}^i) + \overset{2}{S}_{\mu\ \beta\gamma}^{\ \psi}(\bar{\Omega}_{\psi}^{\ \delta}B^i_{\delta} + \bar{\Omega}_{\psi}^{\ \nu}\underset{\nu}{N}^i)$$

In formulas (2.2)-(7.2)

$$\bar{\Omega} = \bar{\Omega}(d_2, d_1)$$

for all indices of $\bar{\Omega}$.

3. A SPECIAL CASE

If the space and its subspace are Riemannian, then from (2.2), (2.3) we obtain (in case the Riemannian space tensors P and S are zero)

$$(3.1)\quad [D_1D_2]R_{j\ hk}^{\ i}B^{jhk}_{\alpha\beta\gamma} + R_{j\ hk}^{\ i}(\bar{\Omega}_{\alpha}^{\ \delta}B^j_{\delta} + \bar{\Omega}_{\alpha}^{\ \nu}\underset{\nu}{N}^j)B^{hk}_{\beta\gamma} +$$
$$+ R_{j\ hk}^{\ i}B^j_{\alpha}(\bar{\Omega}_{\beta}^{\ \delta}B^h_{\delta} + \bar{\Omega}_{\beta}^{\ \nu}\underset{\nu}{N}^h)B^k_{\gamma} + R_{j\ hk}^{\ i}B^{jh}_{\alpha\beta}(\bar{\Omega}_{\gamma}^{\ \delta}B^k_{\delta} + \bar{\Omega}_{\gamma}^{\ \nu}\underset{\nu}{N}^k) =$$
$$= \overset{2}{R}_{\alpha\ ..\beta\gamma}^{\ \varepsilon}(\bar{\Omega}_{\varepsilon}^{\ \delta}B^i_{\delta} + \bar{\Omega}_{\varepsilon}^{\ \nu}\underset{\nu}{N}^i) + \overset{2}{R}_{\alpha\ \beta\gamma}^{\ \mu}(\bar{\Omega}_{\mu}^{\ \delta}B^i_{\delta} + \bar{\Omega}_{\mu}^{\ \nu}\underset{\nu}{N}^i)$$

$$(3.2)\quad [D_1D_2]R_{j\ hk}^{\ i}\underset{\mu}{N}^j B^{hk}_{\beta\gamma} + R_{j\ hk}^{\ i}(\bar{\Omega}_{\mu}^{\ \delta}B^j_{\delta} + \bar{\Omega}_{\mu}^{\ \nu}\underset{\nu}{N}^j)B^{hk}_{\beta\gamma} +$$
$$+ R_{j\ hk}^{\ i}\underset{\mu}{N}^j(\bar{\Omega}_{\beta}^{\ \delta}B^k_{\delta} + \bar{\Omega}_{\beta}^{\ \nu}\underset{\nu}{N}^k)B^k_{\gamma} + R_{j\ hk}^{\ i}\underset{\mu}{N}^j B^k_{\beta}(\bar{\Omega}_{\gamma}^{\ \delta}B^k_{\delta} + \bar{\Omega}_{\gamma}^{\ \nu}\underset{\nu}{N}^k) =$$
$$= \overset{2}{R}_{\mu\ \beta\gamma}^{\ \varepsilon}(\bar{\Omega}_{\varepsilon}^{\ \delta}B^i_{\delta} + \bar{\Omega}_{\varepsilon}^{\ \nu}\underset{\nu}{N}^i) + \overset{2}{R}_{\mu\ \beta\gamma}^{\ \nu}(\bar{\Omega}_{\nu}^{\ \delta}B^i_{\delta} + \bar{\Omega}_{\nu}^{\ \psi}\underset{\psi}{N}^i)$$

In the two above formulas

$$\bar{\Omega}_{\beta}^{\ \delta} = \frac{1}{2}\ \overset{2}{R}_{\beta\ \gamma\delta}^{\ \delta}[d_2u^{\gamma}, d_1u^{\delta}]$$
$$\bar{\Omega}_{\alpha}^{\ \mu} = \frac{1}{2}\ \overset{2}{R}_{\alpha\ \beta\gamma}^{\ \mu}[d_2u^{\beta}, d_1u^{\gamma}]$$
$$\bar{\Omega}_{\mu}^{\ \nu} = \frac{1}{2}\ \overset{2}{R}_{\mu\ \beta\gamma}^{\ \nu}[d_2u^{\beta}, d_1u^{\gamma}]$$
$$\overset{2}{R}_{\alpha\ \beta\gamma}^{\ \mu} = \bar{R}_{\alpha\ \beta\gamma}^{\ \mu} + \bar{\Theta}_{\alpha}^{\ \mu}{}_{[\beta}\ \bar{\Theta}_{|\mu|}{}^{\varepsilon}{}_{\gamma]}$$

$$\overset{2}{R}_{\alpha}{}^{\mu}{}_{\beta\gamma} = \bar{R}_{\alpha}{}^{\mu}{}_{\beta\gamma} + \bar{\Theta}_{\alpha}{}^{\nu}{}_{[\beta}\ \bar{\lambda}_{|\nu|}{}^{\mu}{}_{\gamma]}$$

$$\overset{2}{R}_{\mu}{}^{\nu}{}_{\beta\gamma} = \bar{R}_{\mu}{}^{\nu}{}_{\beta\gamma} + \bar{\lambda}_{\mu}{}^{\xi}{}_{[\beta}\ \bar{\lambda}_{|\xi|}{}^{\nu}{}_{\gamma]}$$

$$\bar{R}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma} = \partial_{[\gamma}T_{|\alpha|}{}^{\varepsilon}{}_{\beta]} + \bar{\Gamma}_{\alpha}{}^{\kappa}{}_{[\beta}\ \bar{T}_{|\kappa|}{}^{\varepsilon}{}_{\gamma]}$$

$$\bar{R}_{\alpha}{}^{\mu}{}_{\beta\gamma} = \partial_{[\gamma}\ \bar{\Theta}_{|\alpha|}{}^{\mu}{}_{\beta]} + \bar{\Gamma}_{\alpha}{}^{\kappa}{}_{[\beta}\ \bar{\Theta}_{|\kappa|}{}^{\mu}{}_{\gamma]}$$

$$\bar{R}_{\mu}{}^{\nu}{}_{\beta\gamma} = \partial_{[\gamma}\ \bar{\lambda}_{|\mu|}{}^{\nu}{}_{\beta]} + \bar{\Theta}_{\mu}{}^{\kappa}{}_{[\beta}\ \bar{\Theta}_{|\kappa|}{}^{\nu}{}_{\gamma]}$$

$$\bar{\Gamma}_{\alpha\gamma\beta} = g_{ir}\ B^{r}_{\gamma}(B^{i}_{\alpha\beta} + \Gamma_{j}{}^{i}{}_{k}\ B^{jk}_{\alpha\beta})$$

$$\bar{\Theta}_{\alpha\nu\beta} = g_{ir}\underset{\nu}{N}^{r}(B^{i}_{\alpha\beta} + \Gamma_{j}{}^{i}{}_{k}B^{jk}_{\alpha\beta})$$

$$\bar{\lambda}_{\mu}{}^{\psi}{}_{\gamma} = \overset{\psi}{N}_{i}\ (\partial_{\gamma}\underset{\mu}{N}^{i} + \Gamma_{j}{}^{i}{}_{k}\underset{\mu}{N}^{j}B^{k}_{\gamma})$$

Multiplying (3.1) and (3.2) with B^{κ}_{i} and $\overset{\rho}{N}_{i}$ we obtain

$$(3.3)\quad ([D_1D_2]R_{j}{}^{i}{}_{hk})B^{j\kappa}_{\alpha i}B^{hk}_{\beta\gamma} + R_{j}{}^{i}{}_{hk}(\bar{\Omega}_{\alpha}{}^{\delta}B^{j}_{\delta} + \bar{\Omega}_{\alpha}{}^{\nu}\underset{\nu}{N}^{j})B^{hk}_{\beta\gamma}\ B^{\kappa}_{i} +$$
$$+ R_{j}{}^{i}{}_{hk}B^{j\kappa}_{\alpha i}\ (\bar{\Omega}_{\beta}{}^{\delta}B^{h}_{\delta}+\bar{\Omega}_{\beta}{}^{\nu}\underset{\nu}{N}^{h})B^{k}_{\gamma} + R_{j}{}^{i}{}_{hk}B^{j\kappa}_{\alpha i}B^{h}_{\beta}(\bar{\Omega}_{\gamma}{}^{\delta}B^{k}_{\delta} + \bar{\Omega}_{\gamma}{}^{\nu}\ \underset{\nu}{N}^{k}) =$$
$$= \overset{2}{R}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\ \bar{\Omega}_{\varepsilon}{}^{\kappa} + \overset{2}{R}_{\alpha}{}^{\mu}{}_{\beta\gamma}\ \bar{\Omega}_{\mu}{}^{\kappa}$$

$$(3.4)\quad ([D_1D_2]R_{j}{}^{i}{}_{hk})B^{j}_{\alpha}\ \overset{\rho}{N}_{i}\ B^{hk}_{\beta\gamma} + R_{j}{}^{i}{}_{hk}(\bar{\Omega}_{\alpha}{}^{\delta}B^{j}_{\delta} + \bar{\Omega}_{\alpha}{}^{\nu}\underset{\nu}{N}^{j})\overset{\rho}{N}_{i}B^{hk}_{\beta\gamma} +$$
$$+ R_{j}{}^{i}{}_{hk}\ B^{j}_{\alpha}\overset{\rho}{N}_{i}(\bar{\Omega}_{\alpha}{}^{\delta}B^{h}_{\delta} + \bar{\Omega}_{\beta}{}^{\nu}\underset{\nu}{N}^{h})B^{k}_{\gamma} + R_{j}{}^{i}{}_{hk}B^{j}_{\alpha}\overset{\rho}{N}_{i}B^{h}_{\beta}(\bar{\Omega}_{\gamma}{}^{\delta}B^{k}_{\delta} + \bar{\Omega}_{\gamma}{}^{\nu}\underset{\nu}{N}^{k}) =$$
$$= \overset{2}{R}_{\alpha}{}^{\varepsilon}{}_{\beta\gamma}\ \bar{\Omega}_{\varepsilon}{}^{\rho} + \overset{2}{R}_{\alpha}{}^{\mu}{}_{\beta\gamma}\ \bar{\Omega}_{\mu}{}^{\rho}.$$

$$(3.5)\quad ([D_1D_2]R_{j}{}^{i}{}_{hk})\underset{\mu}{N}^{j}B^{\kappa}_{i}B^{hk}_{\beta\gamma} + R_{j}{}^{i}{}_{hk}(\bar{\Omega}_{\mu}{}^{\delta}B^{j}_{\delta} + \bar{\Omega}_{\mu}{}^{\nu}\underset{\nu}{N}^{j})B^{\kappa}_{i}P^{hk}_{\beta\gamma} +$$
$$+ R_{j}{}^{i}{}_{hk}\underset{\mu}{N}^{i}B^{\kappa}_{i}(\bar{\Omega}_{\beta}{}^{\delta}B^{h}_{\delta} + \bar{\Omega}_{\beta}{}^{\nu}\underset{\nu}{N}^{h})B^{k}_{\gamma} + R_{j}{}^{i}{}_{hk}\underset{\mu}{N}^{j}B^{\kappa}_{i}B^{h}_{\beta}(\bar{\Omega}_{\gamma}{}^{\delta}B^{k}_{\delta} + \bar{\Omega}_{\gamma}{}^{\nu}\underset{\nu}{N}^{k}) =$$
$$= \overset{2}{R}_{\mu}{}^{\varepsilon}{}_{\beta\gamma}\ \bar{\Omega}_{\varepsilon}{}^{\kappa} + \overset{2}{R}_{\mu}{}^{\nu}{}_{\beta\gamma}\ \bar{\Omega}_{\nu}{}^{\kappa}\ .$$

$$(3.6)\quad ([D_1D_2]R_{j}{}^{i}{}_{hk})\underset{\mu}{N}^{j}\overset{\rho}{N}_{i}B^{hk}_{\beta\gamma} + R_{j}{}^{i}{}_{hk}(\bar{\Omega}_{\mu}{}^{\delta}B^{j}_{\delta} + \bar{\Omega}_{\mu}{}^{\nu}\underset{\nu}{N}^{j})\overset{\rho}{N}_{i}B^{hk}_{\beta\gamma} +$$
$$+ R_{j}{}^{i}{}_{hk}\underset{\mu}{N}^{j}\overset{\rho}{N}_{i}(\bar{\Omega}_{\beta}{}^{\delta}B^{k}_{\delta} + \bar{\Omega}_{\beta}{}^{\nu}\underset{\nu}{N}^{h})B^{k}_{\gamma} + R_{j}{}^{i}{}_{hk}\underset{\mu}{N}^{j}\overset{\rho}{N}_{i}B^{h}_{\beta}(\bar{\Omega}_{\gamma}{}^{\delta}B^{k}_{\delta} + \bar{\Omega}_{\gamma}{}^{\nu}\underset{\nu}{N}^{k}) =$$
$$= \overset{2}{R}_{\mu}{}^{\varepsilon}{}_{\beta\gamma}\ \bar{\Omega}_{\varepsilon}{}^{\rho} + \overset{2}{R}_{\mu}{}^{\nu}{}_{\beta\gamma}\ \bar{\Omega}_{\nu}{}^{\rho}$$

Let us define the tensors

$$
\begin{array}{ll}
\overset{2}{R}{}_{\alpha}{}^{\kappa}{}_{\beta\gamma} = R_j{}^{i}{}_{hk} B^{j}_{\alpha} B^{\kappa}_{i} B^{hk}_{\beta\gamma} & \overset{2}{R}{}_{\alpha}{}^{\mu}{}_{\beta\gamma} = R_j{}^{i}{}_{hk} B^{j}_{\alpha} N^{\mu}_{i} B^{hk}_{\beta\gamma} \\
\overset{2}{R}{}_{\mu}{}^{\kappa}{}_{\beta\gamma} = R_j{}^{i}{}_{hk} N^{j}_{\mu} B^{\kappa}_{i} B^{hk}_{\beta\gamma} & \overset{2}{R}{}_{\alpha}{}^{\kappa}{}_{\mu\gamma} = R_j{}^{i}{}_{hk} B^{j}_{\alpha} B^{\kappa}_{i} N^{k}_{\mu} B^{k}_{\gamma} \\
\overset{2}{R}{}_{\alpha}{}^{\kappa}{}_{\beta\mu} = R_j{}^{i}{}_{hk} B^{j}_{\alpha} B^{\kappa}_{i} B^{h}_{\beta} N^{k}_{\mu} & \overset{2}{R}{}_{\nu}{}^{\mu}{}_{\beta\gamma} = R_j{}^{i}{}_{hk} N^{j}_{\nu} N^{\mu}_{i} B^{hk}_{\beta\gamma} \\
\overset{2}{R}{}_{\alpha}{}^{\mu}{}_{\nu\gamma} = R_j{}^{i}{}_{hk} B^{j}_{\alpha} N^{\mu}_{i} N^{h}_{\nu} B^{k}_{\gamma} & \overset{2}{R}{}_{\alpha}{}^{\mu}{}_{\beta\nu} = R_j{}^{i}{}_{hk} B^{j}_{\alpha} N^{\mu}_{i} B^{h}_{\beta} N^{k}_{\nu} \\
\overset{2}{R}{}_{\mu}{}^{\nu}{}_{\psi\gamma} = R_j{}^{i}{}_{hk} N^{j}_{\mu} N^{\nu}_{i} N^{h}_{\psi} B^{k}_{\gamma} & \overset{2}{R}{}_{\mu}{}^{\nu}{}_{\beta\psi} = R_j{}^{i}{}_{hk} N^{j}_{\mu} N^{\nu}_{i} B^{h}_{\beta\psi} N^{k}
\end{array}
\tag{3.7}
$$

Some of these tensors appear in (1.5) and (1.6) and for the Riemannian space and subspace are the same as those (above) defined.

Now (3.3) - (3.6) have the from

$$
\begin{aligned}
&([D_1 D_2] R_j{}^{i}{}_{hk}) B^{j}_{\alpha} B^{\kappa}_{i} B^{hk}_{\beta\gamma} = \\
&= \overset{2}{R}{}_{\alpha}{}^{\delta}{}_{\beta\gamma} \bar{\Omega}_{\delta}{}^{\kappa} - \overset{2}{R}{}_{\delta}{}^{\kappa}{}_{\beta\gamma} \bar{\Omega}_{\alpha}{}^{\gamma} - \overset{2}{R}{}_{\alpha}{}^{\kappa}{}_{\delta\gamma} \bar{\Omega}_{\beta}{}^{\delta} - \overset{2}{R}{}_{\alpha}{}^{\kappa}{}_{\beta\delta} \bar{\Omega}_{\gamma}{}^{\delta} + \\
&+ \overset{2}{R}{}_{\alpha}{}^{\mu}{}_{\beta\gamma} \bar{\Omega}_{\mu}{}^{\kappa} - \overset{2}{R}{}_{\mu}{}^{\kappa}{}_{\beta\gamma} \bar{\Omega}_{\alpha}{}^{\mu} - \overset{2}{R}{}_{\alpha}{}^{\kappa}{}_{\mu\gamma} \bar{\Omega}_{\beta}{}^{\mu} - \overset{2}{R}{}_{\alpha}{}^{\kappa}{}_{\beta\mu} \bar{\Omega}_{\gamma}{}^{\mu} \overset{\text{def}}{=\!=} [\bar{D}_1 \bar{D}_2] \overset{2}{R}{}_{\alpha}{}^{\kappa}{}_{\beta\gamma}
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
&([D_1 D_2] R_j{}^{i}{}_{hk}) B^{j}_{\alpha} N^{\nu}_{i} B^{hk}_{\beta\gamma} = \\
&= \overset{2}{R}{}_{\alpha}{}^{\delta}{}_{\beta\gamma} \bar{\Omega}_{\delta}{}^{\nu} - \overset{2}{R}{}_{\delta}{}^{\nu}{}_{\beta\gamma} \bar{\Omega}_{\alpha}{}^{\delta} - \overset{2}{R}{}_{\alpha}{}^{\beta}{}_{\delta\gamma} \bar{\Omega}_{\beta}{}^{\delta} - \overset{2}{R}{}_{\alpha}{}^{\nu}{}_{\beta\delta} \bar{\Omega}_{\gamma}{}^{\delta} + \\
&+ \overset{2}{R}{}_{\alpha}{}^{\mu}{}_{\beta\gamma} \bar{\Omega}_{\mu}{}^{\nu} - \overset{2}{R}{}_{\mu}{}^{\rho}{}_{\beta\gamma} \bar{\Omega}_{\alpha}{}^{\mu} - \overset{2}{R}{}_{\alpha}{}^{\nu}{}_{\beta\mu} \bar{\Omega}_{\gamma}{}^{\mu} - \overset{2}{R}{}_{\alpha}{}^{\nu}{}_{\beta\mu} \bar{\Omega}_{\gamma}{}^{\mu} \overset{\text{def}}{=\!=} [\bar{D}_1 \bar{D}_2] \overset{2}{R}{}_{\alpha}{}^{\nu}
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&([D_1 D_2] R_j{}^{i}{}_{hk}) N^{j}_{\mu} B^{\kappa}_{i} B^{hk}_{\beta\gamma} = \\
&= \overset{2}{R}{}_{\mu}{}^{\delta}{}_{\beta\gamma} \bar{\Omega}_{\delta}{}^{\kappa} - \overset{2}{R}{}_{\delta}{}^{\kappa}{}_{\beta\gamma} \bar{\Omega}_{\mu}{}^{\delta} - \overset{2}{R}{}_{\mu}{}^{\kappa}{}_{\delta\gamma} \bar{\Omega}_{\beta}{}^{\delta} - \overset{2}{R}{}_{\mu}{}^{\kappa}{}_{\beta\delta} \bar{\Omega}_{\gamma}{}^{\delta} + \\
&+ \overset{2}{R}{}_{\mu}{}^{\nu}{}_{\beta\gamma} \bar{\Omega}_{\nu}{}^{\kappa} - \overset{2}{R}{}_{\nu}{}^{\kappa}{}_{\beta\gamma} \bar{\Omega}_{\mu}{}^{\nu} - \overset{2}{R}{}_{\mu}{}^{\kappa}{}_{\nu\gamma} \bar{\Omega}_{\beta}{}^{\nu} - \overset{2}{R}{}_{\mu}{}^{\kappa}{}_{\beta\nu} \bar{\Omega}_{\gamma}{}^{\nu} \overset{\text{def}}{=\!=} [\bar{D}_1 \bar{D}_2] \overset{2}{R}{}_{\mu}{}^{\kappa}{}_{\beta\gamma} .
\end{aligned}
\tag{3.10}
$$

(3.11) $$([D_1 D_2] R_j{}^i{}_{hk} \, N^j_\mu \, \overset{\nu}{N}_i \, B^{hk}_{\beta\gamma} = \\ = \overset{2}{\bar{R}}_\mu{}^\delta{}_{\beta\gamma} \bar{\Omega}_\delta{}^\nu - \overset{2}{\bar{R}}_\delta{}^\nu{}_{\beta\gamma} \bar{\Omega}_\mu{}^\delta - \overset{2}{\bar{R}}_\mu{}^\nu{}_{\delta\gamma} \bar{\Omega}_\beta{}^\delta - \overset{2}{\bar{R}}_\mu{}^\nu{}_{\beta\delta} \bar{\Omega}_\gamma{}^\delta + \\ + \overset{2}{\bar{R}}_\mu{}^\psi{}_{\beta\gamma} \bar{\Omega}_\psi{}^\nu - \overset{2}{\bar{R}}_\psi{}^\nu{}_{\beta\gamma} \bar{\Omega}_\mu{}^\psi - \overset{2}{\bar{R}}_\mu{}^\nu{}_{\psi\gamma} \bar{\Omega}_\beta{}^\psi - \overset{2}{\bar{R}}_\mu{}^\nu{}_{\beta\psi} \bar{\Omega}_\gamma{}^\psi \overset{def}{\equiv} \\ [\bar{D}_1 \bar{D}_2] \overset{2}{\bar{R}}_\mu{}^\nu{}_{\beta\gamma} \,.$$

In the Riemannian space the double alternated differentials of the curvature tensor of the space and its subspace are connected by (3.8)-(3.11).

4. RECURRENT RIEMANNIAN SPACE OF THE SECOND ORDER

If the surrounding Riemannian space has the property

(4.1) $$R_j{}^i{}_{hk|p|q} = a_{pq} \, R_j{}^i{}_{hk}$$

then from

$$[D_1 D_2] R_j{}^i{}_{hk} = R_j{}^i{}_{hk[|p|q]} \, [d_2 x^p, d_1 x^q]$$

it follows that

(4.2) $$[D_1 D_2] R_j{}^i{}_{hk} = \frac{1}{2}(a_{pq} - a_{qp}) R_j{}^i{}_{hk} [d_2 x^p, d_1 x^q] = \\ = b_{pq} \, R_j{}^i{}_{hk} \, [d_2 x^p, d_1 x^q]$$

where b_{pq} is the second order antisymmetric covariant tensor

$$b_{pq} = \frac{1}{2} \, (a_{pq} - a_{qp})$$

In this case (3.8) becomes

(4.3) $$[\bar{D}_1 \bar{D}_2] \; \overset{2}{\bar{R}}_\alpha{}^\kappa{}_{\beta\gamma} = b_{pq} [d_2 x^p, d_1 x^q] \; R_j{}^i{}_{hk} \; B^{j\,\kappa h k}_{\alpha\, i\, \beta\gamma}$$

Denoting

$$K = b_{pq} \; [d_2 x^p, \; d_1 x^q]$$

we have for (4.3)

$$(4.4) \qquad [\bar{D}_1\bar{D}_2]\ \overset{2}{R}_{\alpha}{}^{\kappa}{}_{\beta\gamma} = K R_j{}^i{}_{hk}\ B^{j\ \kappa\ h\ k}_{\alpha\ \ \beta\ \gamma} = K \overset{2}{R}_{\alpha}{}^{\kappa}{}_{\beta\gamma}$$

In a similar manner (3.9), (3.10) and (3.11) become in this case

$$(4.5) \qquad [\bar{D}_1\bar{D}_2]\ \overset{2}{R}_{\alpha}{}^{\nu}{}_{\beta\gamma} = K \overset{2}{R}_{\alpha}{}^{\nu}{}_{\beta\gamma}$$

$$(4.6) \qquad [\bar{D}_1\bar{D}_2]\ \overset{2}{R}_{\mu}{}^{\cdot\kappa}{}_{\beta\gamma} = K \overset{2}{R}_{\mu}{}^{\kappa}{}_{\beta\gamma}$$

$$(4.7) \qquad [\bar{D}_1\bar{D}_2]\ \overset{2}{R}_{\mu}{}^{\nu}{}_{\beta\gamma} = K \overset{2}{R}_{\mu}{}^{\nu}{}_{\beta\gamma}$$

From the above we can conclude.

If in the Riemannian space the curvature tensor R has the property (4.1) then the curvature tensor $\overset{2}{R}_{\alpha}{}^{\kappa}{}_{\beta\gamma}$, $\overset{2}{R}_{\alpha}{}^{\nu}{}_{\beta\gamma}$, $\overset{2}{R}_{\mu}{}_{\beta\gamma}$, $\overset{2}{R}_{\mu}{}^{\nu}{}_{\beta\gamma}$ defined by (3.7) have the property (4.4)-(4.7), where the left hand side of these formulas are defined by (3.8)-(3.11).

The formulas (4.4)-(4.7) are valid if instead of condition (4.1) we have the weaker condition

$$(4.8) \qquad [D_1D_2]R_j{}^i{}_{hk} = K R_j{}^i{}_{hk}$$

From (4.1) follows (4.8) for every motion d_1x^p, d_2x^q, but from (4.8) only (4.2) follows.

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REZIME

VEZA IZMEDJU DUPLOG ALTERNIRANOG APSOLUTNOG DIFERENCIJALA TENZORA KRIVINE FINSLEROVOG PROSTOPA I INDUKOVANIH KRIVINA PODPROSTORA

U uvodu su date formule (1.15)-(1.20) koje povezuju tenzore krivi na prostoru F_n i potprostora F_m. U 2. su date veze izmedju duplog alterniranog apsolutnog diferencijala ovih tenzora krivina. Te formule se uprošćuju za Riemannov prostor, što je odredjeno u 3. U 4. je ispitivan 2 - rekurentni Riemannov prostor i njegov potprostor. Tada važe formule (4.4) - (4.7).

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CONNEXIONS IN f-MANIFOLDS

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1. K.Yano |6| introduced the notion of an f-structure on a C^∞ manifold N as a tensor field f of type (1,1) and rank 2m $(2m \le n = \dim N)$ satisfying $f^3 + f = 0$, the existence of which is equivalent to a reduction of the structural group of the tangent bundle to the group $Gl(m,C) \times Gl(n-2m,R)$. A manifold N with an f-structure is called an f-manifold. Almost complex $(2m = n)$ and almost contact $(2m = n-1)$ structures are well-known examples of f-structures.

Let N be an n-dimensional manifold with an f-structure of rank 2m. If there exists on N vector fields ξ_α, $\alpha = 1,\dots,n-2m$, such that if η_α are dual 1-forms, then

$$\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta} \tag{1}$$

$$f^2 = -I + \Sigma_\alpha\, \eta_\alpha \otimes \xi_\alpha \tag{2}$$

we say that the f-structure has complemented frames. As an immediate consequence of (1) and (2), we obtain

$$f(\xi_\alpha) = 0$$

$$\eta_\alpha \circ f = 0$$

On a manifold N the existence of an f-structure with complemented frames is equivalent to a reduction of the structural group of the tangent bundle to the group $Gl(m,C) \times E_{n-2m}$, where E_{n-2m} denotes (n-2m) x (n-2m) unit matrix (see |4|).

The purpose of this paper is to consider affine connexions in f-manifolds.

2. Let N be an n-dimensional manifold of class C^∞ with an f-structure of rank 2m. Then the structural group of the tangent bundle can be reduced to the group $Gl(m,C) \times Gl(n-2m,R)$, and conversely.

Let $(fB(N),\ Gl(m,C) \times Gl(n-2m,R), N)$ be the principal bundle of adapted frames of f-structure on N, i.e. the principal bundle of frames with respect to which f has components

$$(3) \qquad f = \begin{bmatrix} 0 & E_m & 0 \\ -E_m & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where E_m denotes $m \times m$ unit matrix. With respect to every connexion on the bundle fB(N) the tensor field f is parallel. Indeed, let $\gamma : [0,1] \to N$ be a curve of class C^∞ on the manifold N, and let $\tilde{\gamma}$ be a horizontal lift of γ in the bundle fB(N) with respect to the given connexion. If we denote $\tilde{\gamma}(0) = (\gamma(0), u_1(0), \dots, u_n(0))$ and $\tilde{\gamma}(t) = (\gamma(t), u_1(t), \dots, u_n(t))$, where $(u_1(0), \dots, u_n(0))$ and $(u_1(t), \dots, u_n(t))$ are any frames from fB(N) at points $\gamma(0)$ and $\gamma(t)$ respectively, then it is obvious that components of f with respect to frames $(u_1(0), \dots, u_n(0))$ and $(u_1(t), \dots, u_n(t))$ are equal and given by (3), which means that tensor field f is parallel along the curve γ.

If H' is a connexion on fB(N), then it can be extended to a connexion H on B(N) by the right action of the group $Gl(n,R)$ (B(N) is bundle of basses of N). Hence, H gives rise to a parallel translation along curves in N. Further, it is clear that with respect to this parallel translation, the tensor field f is parallel. Indeed, let we have parallel translation along a curve γ. We may choose a horizontal lift $\tilde{\gamma}$ of γ so that $\tilde{\gamma}(0) \in fB(N)$. Then we have $\tilde{\gamma} \subset fB(N)$, which follows from the definition of the connexion H.

Conversely, if H is a connexion on B(N), such that the tensor field f is parallel with respect to H, then H comes from a connexion H' on fB(N) in the above manner. Indeed, let $b=(m,u_1,\dots,u_m)\in fB(N)$ and let $\tilde{\gamma}$ be a horizontal curve in B(N) passing through b. Then every point on $\tilde{\gamma}$ must belong to fB(N), since f is parallel along the curve $\gamma=\pi\circ\tilde{\gamma}$ (π is the projection $B(N)\to N$), by assumption. Therefore $H_b\subset T_b(fB(N))$, so we may define H' by $H'_b=H_b$.

Thus we have

THEOREM 1. *For an affine connexion* H *on a manifold* N *with* f*-structure, the following conditions are equivalent:*

(a) H *is the extension of a connexion of* fB(N)

(b) The tensor field f *is parallel with respect to* H .

An affine connexion on N is said to be an f-connexion, if it satisfies any one (and hence both) of the conditions above.

From the general theory of connexions (see |1|) we know thet every principal bundle (P, G, N), with N paracompact, admits a connexion. This means that every paracompact manifold N with an f-structure, admits an f-connecion.

Let the f-structure on N have complemented frames. Then the structural group of the tangent bundle can be reduced to the group $Gl(m,C)\times E_{n-2m}$, and conversely.

Let $((f,\xi_\alpha,\eta_\alpha)B(N),\ Gl(m,C)\times E_{n-2m},N)$ be the principal bundle of adapted frames of f-structure with complemented frames on N, i.e. the principal bundle of frames with respect to which f has components (3), while η_α and ξ_α have components

$$\eta_\alpha = (\underbrace{0,\dots,0}_{2m},\underbrace{0,\dots 1}_{\alpha},\dots 0,)\ \xi_\alpha = \left[\begin{array}{c}0\\ \vdots\\ 0\\ 0\\ \vdots\\ 1\\ \vdots\\ 0\end{array}\right] \begin{array}{l}\left.\begin{array}{c}\\ \\ \\ \end{array}\right\}2m \\ \left.\begin{array}{c}\\ \\ \\ \end{array}\right\}\alpha \\ \\ \\ \end{array}$$

It is obvious, that the tensor field f, vector fields ξ_α and 1-forms η_α are parallel with respect to every connexion in the bundle $(f,\xi_\alpha,\eta_\alpha)B(N)$.

The proof of the following theorem is analogous to that of Theorem 1.

THEOREM 2. *For an affine connexion* H *on a manifold* N *with* f*-structure with complemented frames, the following conditions are equivalent:*

a) H *is the extension of a connexion of* $(f,\xi_\alpha,\eta_\alpha)B(N)$

b) The tensor field f, *vector fields* ξ_α *and* 1*-forms* η_α *are parallel with respect to* H.

An affine connexion on N *is said to be an* $(f,\xi_\alpha,\eta_\alpha)$ *-conexion, if it satisfies any one (and hence both) of the conditions above.*

As a consequence of the above theorem, we have that every paracompact manifold N with an f-structure with complemented frames, admits an $(f,\xi_\alpha,\eta_\alpha)$ -connexion.

We shall now prove the existence of a connexion of a more special type.

THEOREM 3. *Every manifold* N *with an* f*-structure with complemented frames admits an* $(f,\xi_\alpha,\eta_\alpha)$ *-connexion such that its torsion* T *is given by*

$$4T(X,Y) = 4\,\Sigma_\alpha d\eta_\alpha(X,Y)\xi_\alpha - \Sigma_\alpha \eta_\alpha([f,f](X,Y))\xi_\alpha + \\ + [f,f](X + \Sigma_\alpha \eta_\alpha(X)\xi_\alpha, Y + \Sigma_\beta \eta_\beta(Y)\xi_\beta)\ ,$$

where $[f,f]$ *is the Nijenhuis torsion of* f, *and* $X,Y \in \mathfrak{X}(N)$ ($\mathfrak{X}(N)$ *is the set of all vector fields of class* C^∞ *on* N*).*

P r o o f. Consider an arbitrary symmetric affine connexion on N (if N is paracompact, such a connexion exists) with covariant differentiation B. Let ∇ be the covariant differention with respect to a desired $(f,\xi_\alpha,\eta_\alpha)$ -connexion, and let $X,Y \in \mathfrak{X}(N)$ Then

$$\nabla_X \xi_\alpha = 0, \quad \text{for any } \alpha$$

$$\nabla_X \eta_\alpha = 0, \quad \text{for any } \alpha$$

$$\nabla_X f = 0. \tag{4}$$

From (4) we obtain

$$\nabla_X (fY) = f\nabla_X Y . \tag{5}$$

It may be written

$$\nabla_X Y = B_X Y - H(X,Y) \tag{6}$$

where H is a tensor field of type (1,2). Now we have

$$H(X,\xi_\alpha) = B_X \xi_\alpha \tag{7}$$

$$\eta_\alpha(H(X,Y)) = \eta_\alpha(B_X Y) - X\eta_\alpha(Y). \tag{8}$$

From (5) and (6) we obtain

$$H(X,fY) - fH(X,Y) = B_X(fY) - fB_X Y .$$

After applying f to this equation and to Y, we obtain

$$fH(X,f^2Y) - f^2H(X,fY) = fB_X(f^2Y) - f^2B_X(fY).$$

Since by applying f to this equation and to Y again we have the same equation, the general solution of this equation is given by

$$4fH(X,f^2Y) = 2fB_X(f^2Y) - 2f^2B_X(fY) - fW(X,fY) + f^2W(X,f^2Y) ,$$

where W is a tensor field of type (1,2). From this equation after applying f, and from (2),(7) and (8), we obtain

$$4H(X,Y) = 4\Sigma_\alpha\,(\eta_\alpha(B_XY) - X\eta_\alpha(Y))\xi_\alpha - 4\,\Sigma_\alpha\eta_\alpha(Y)f^2B_X\xi_\alpha +$$
$$+ 2f^2(B_Xf)(fY) - f^2W(X,fY) - fW(X,f^2Y).$$

Let W be defined by

$$W(X,Y) = (B_Yf)X - \Sigma_\alpha\eta_\alpha(X)B_{fY}\xi_\alpha.$$

Then we have

$$\begin{aligned}
4T(X,Y) &= 4\nabla_XY - 4\nabla_YX - 4[X,Y]\\
&= 4B_XY - 4B_YX - 4[X,Y] - 4H(X,Y) + 4H(Y,X)\\
&= -4\Sigma_\alpha(\eta_\alpha(B_XY) - X\eta_\alpha(Y))\xi_\alpha + 4\Sigma_\alpha\eta_\alpha(Y)f^2B_X\xi_\alpha -\\
&- 2f^2(B_Xf)(fY) + f^2(B_{fY}f)X - \Sigma_\alpha\eta_\alpha(X)f^2B_{f^2Y}\xi_\alpha +\\
&+ f(B_{f^2Y}f)X + \Sigma_\alpha\eta_\alpha(X)fB_{fY}\xi_\alpha +\\
&+ 4\Sigma_\alpha(\eta_\alpha(B_YX) - Y\eta_\alpha(X))\xi_\alpha - 4\Sigma_\alpha\eta_\alpha(X)f^2B_Y\xi_\alpha +\\
&+ 2f^2(B_Yf)(fX) - f^2(B_{fX}f)Y + \Sigma_\alpha\eta_\alpha(Y)f^2B_{f^2X}\xi_\alpha -\\
&- f(B_{f^2X}f)Y - \Sigma_\alpha\eta_\alpha(Y)fB_{fX}\xi_\alpha\\
&= 4\Sigma_\alpha d\eta_\alpha(X,Y)\xi_\alpha + 4\Sigma_\alpha\eta_\alpha(Y)f^2B_X\xi_\alpha -\\
&- 4\Sigma_\alpha\eta_\alpha(X)f^2B_Y\xi_\alpha - 2f^2B_X(f^2Y) -\\
&- 2fB_X(fY) + 2f^2B_Y(f^2X) + 2fB_Y(fX) +\\
&+ f^2B_{fY}(fX) + fB_{fY}X + fB_{f^2Y}(fX) -\\
&- f^2B_{f^2Y}X - f^2B_{fX}(fY) - fB_{fX}Y - fB_{f^2X}(fY) +\\
&+ f^2B_{f^2X}Y + \Sigma_\alpha\eta_\alpha(X)f^2B_Y\,\xi_\alpha -\\
&- \Sigma_\alpha\eta_\alpha(X)f^2B_{\Sigma_\beta\eta_\beta(Y)\xi_\beta}\xi_\alpha + \Sigma_\alpha\eta_\alpha(X)fB_{fY}\xi_\alpha -\\
&- \Sigma_\alpha\eta_\alpha(Y)f^2B_X\xi_\alpha + \Sigma_\alpha\eta_\alpha(Y)f^2B_{\Sigma_\beta\eta_\beta(X)\xi_\beta}\xi_\alpha -\\
&- \Sigma_\alpha\eta_\alpha(Y)fB_{fX}\,\xi_\alpha.
\end{aligned}$$

Since it may be written

$$-2f^2B_X(f^2Y) = 2f^2B_XY - 2f^2B_X(\Sigma_\alpha\eta_\alpha(Y)\xi_\alpha)$$
$$= 2f^2B_XY - 2\Sigma_\alpha\eta_\alpha(Y)f^2B_X\xi_\alpha$$
$$2f^2B_Y(f^2X) = -2f^2B_YX + 2\Sigma_\alpha\eta_\alpha(X)f^2B_Y\xi_\alpha,$$

we have

$$\begin{aligned}
4T(X,Y) &= 4\Sigma_\alpha d\eta_\alpha(X,Y)\xi_\alpha + \Sigma_\alpha\eta_\alpha(Y)f^2B_X\xi_\alpha - \Sigma_\alpha\eta_\alpha(X)f^2B_Y\xi_\alpha + \\
&+ 2f^2B_XY - 2f^2B_YX - 2f_BX(fY) + 2fB_Y(fX) - f^2[fX,fY] + \\
&+ fB_{fY}X - fB_{fX}Y - fB_Y(fX) + \Sigma_\alpha\eta_\alpha(Y)fB_{\xi_\alpha}(fX) + \\
&+ f^2B_YX - \Sigma_\alpha\eta_\alpha(Y)f^2B_{\xi_\alpha}X + fB_X(fY) - \\
&- \Sigma_\alpha\eta_\alpha(X)fB_{\xi_\alpha}(fY) - f^2B_XY + \Sigma_\alpha\eta_\alpha(X)f^2B_{\xi_\alpha}Y + \\
&+ \Sigma_\alpha\Sigma_\beta\eta_\alpha(X)\eta_\beta(Y)f^2(B_{\xi_\alpha}\xi_\beta - B_{\xi_\beta}\xi_\alpha) + \\
&+ \Sigma_\alpha\eta_\alpha(X)fB_{fY}\xi_\alpha - \Sigma_\alpha\eta_\alpha(Y)fB_{fX}\xi_\alpha \\
&= 4\Sigma_\alpha d\eta_\alpha(X,Y)\xi_\alpha + \Sigma_\beta\eta_\beta(Y)f^2[X,\xi_\beta] + \\
&+ \Sigma_\alpha\eta_\alpha(X)f^2[\xi_\alpha,Y] - \Sigma_\beta\eta_\beta(Y)f[fX,\xi_\beta] - \\
&- \Sigma_\alpha\eta_\alpha(X)f[\xi_\alpha,fY] + \Sigma_\alpha\Sigma_\beta\eta_\alpha(X)\eta_\beta(Y)f^2[\xi_\alpha,\xi_\beta] - \\
&- f^2[fX,fY] + f^2[X,Y] - f[X,fY] - f[fX,Y] = \\
&= 4\Sigma_\alpha d\eta_\alpha(X,Y)\xi_\alpha - f^2([fX,fY] - f[X,fY] - f[fX,Y] + \\
&+ f^2[X,Y]) + \Sigma_\beta\eta_\beta(Y)[f,f](X,\xi_\beta) + \\
&+ \Sigma_\alpha\eta_\alpha(X)[f,f](\xi_\alpha,Y) + \Sigma_\alpha\Sigma_\beta\eta_\alpha(X)\eta_\beta(Y)[f,f](\xi_\alpha,\xi_\beta) \\
&= 4\Sigma_\alpha d\eta_\alpha(X,Y)\xi_\alpha - f^2[f,f](X,Y) + \\
&+ [f,f](X,\Sigma_\beta\eta_\beta(Y)\xi_\beta) + [f,f](\Sigma_\alpha\eta_\alpha(X)\xi_\alpha,Y) + \\
&+ [f,f](\Sigma_\alpha\eta_\alpha(X)\xi_\alpha,\Sigma_\beta\eta_\beta(Y)\xi_\beta) = \\
&= 4\Sigma_\alpha d\eta_\alpha(X,Y)\xi_\alpha - \Sigma_\alpha\eta_\alpha([f,f](X,Y))\xi_\alpha + \\
&+ [f,f](X,Y + \Sigma_\beta\eta_\beta(Y)\xi_\beta) + [f,f](\Sigma_\alpha\eta_\alpha(X)\xi_\alpha,Y + \\
&+ \Sigma_\beta\eta_\beta(Y)\xi_\beta) = \\
&= 4\Sigma_\alpha d\eta_\alpha(X,Y)\xi_\alpha - \Sigma_\alpha\eta_\alpha([f,f](X,Y))\xi_\alpha + \\
&+ [f,f](X + \Sigma_\alpha\eta_\alpha(X)\xi_\alpha, Y + \Sigma_\beta\eta_\beta(Y)\xi_\beta) .
\end{aligned}$$

Now we can prove the next theorem.

THEOREM 4. *A manifold* N *with an* f-*structure with complemented frames admits a symmetric* $(f,\xi_\alpha,\eta_\alpha)$-*connexion, if and only if*

$$[f,f] = 0 \tag{9}$$

$$[f,f] + \Sigma_\alpha d\eta_\alpha \otimes \xi_\alpha = 0 \tag{10}$$

(If an f-*structure with complemented frames satisfies (10), we say it is normal).*

P r o o f. (9) and (10) imply $d\eta_\alpha = 0$, so that Theorem 4 is a special case of Theorem 3, i.e. the $(f,\xi_\alpha,\eta_\alpha)$ -connexion from Theorem 3 has the torsion $T = 0$.

Assume that N admits a symmetric $(f,\xi_\alpha,\eta_\alpha)$ -connexion and denote its covariant differentiation by ∇. Then, for some $X,Y \in \mathfrak{X}(N)$, we obtain

$$\begin{aligned}[f,f](X,Y) &= [fX,fY] - f[fX,Y] - f[X,fY] + f^2[X,Y] \\ &= \nabla_{fX}(fY) - \nabla_{fY}(fX) - f\nabla_{fX}Y + f\nabla_Y(fX) - \\ &\quad - f\nabla_X(fY) + f\nabla_{fY}X + f^2\nabla_XY - f^2\nabla_YX \\ &= (\nabla_{fX}f)Y - (\nabla_{fY}f)X + f(\nabla_Yf)X - f(\nabla_Xf)Y = 0\end{aligned}$$

$$d\eta_\alpha(X,Y) = X\eta_\alpha(Y) - Y\eta_\alpha(X) - \eta_\alpha([X,Y])$$

$$\begin{aligned}d\eta_\alpha(\xi_\beta,\xi_\gamma) &= \xi_\beta\eta_\alpha(\xi_\gamma) - \xi_\gamma\eta_\alpha(\xi_\beta) - \eta_\alpha([\xi_\beta,\xi_\gamma]) \\ &= \eta_\alpha(\nabla_{\xi_\beta}\xi_\gamma - \nabla_{\xi_\gamma}\xi_\beta) = 0\end{aligned}$$

$$\begin{aligned}d\eta_\alpha(-f^2X,\xi_\gamma) &= (-f^2X)\,\eta_\alpha(\xi_\gamma) - \xi_\gamma\eta_\alpha(-f^2X) - \xi_\alpha([-f^2X,\xi_\gamma]) \\ &= -\eta_\alpha(\nabla_{-f^2X}\xi_\gamma - \nabla_{\xi_\gamma}(-f^2X)) \\ &= -\eta_\alpha(\nabla_{\xi_\gamma}\ (f(fX))) \\ &= -\eta_\alpha((\nabla_{\xi_\gamma}f)(fX)) - \eta_\alpha(f(\nabla_{\xi_\gamma}(fX))) = 0\end{aligned}$$

$$d\eta_\alpha(-f^2X,-f^2Y) = (-f^2X)\eta_\alpha(-f^2Y) - (f^2Y)\eta_\alpha(-f^2X) - \\ - \eta_\alpha([-f^2X, -f^2Y]) = \\ = -\eta_\alpha([f(fX), f(fY)]) \\ = -\eta_\alpha(f[f^2X,fY] + f[fX,f^2Y] - f^2[fX,fY] + \\ + [f,f](fX,fY)) = 0$$

For any $X \in \mathfrak{X}(N)$ we have $X = -f^2X + \Sigma_\alpha \eta_\alpha(X)\xi_\alpha$, so that we have $d\eta_\alpha(X,Y) = 0$ for any $X,Y \in \mathfrak{X}(N)$.

THEOREM 5. *Let* N *be a manifold with an* f*-structure. Then the torsion* T *and the curvature* R *of an* f*-connexion satisfy the following identities:*

(1)′ $$T(fX,fY) - fT(fX, Y) - fT(X,fY) + f^2T(X,Y) = \\ = -[f,f](X,Y) ,$$

(2)′ $$R(X,Y) \circ f = f \circ R(X,Y) ,$$

where $X,Y \in \mathfrak{X}(N)$. *If the* f*-structure has complemented frames, then the torsion* T *and the curvature* R *of an* $(f,\xi_\alpha,\eta_\alpha)$*-connexion satisfy some more identities:*

(3)′ $$\eta_\alpha \circ T = d$$

(4)′ $$R(X,Y)\xi_\alpha = 0$$

(5)′ $$\eta_\alpha \circ R(X,Y) = 0$$

where $X,Y \in \mathfrak{X}(N)$.

P r o o f. Let $X,Y,Z \in \mathfrak{X}(N)$. Then we have

(1)′ Follows from $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

(2)′ Follows from $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$

(3)′ $$(\eta_\alpha \circ T)(X,Y) = \eta_\alpha(\nabla_X Y) - \eta_\alpha(\nabla_Y X) - \eta_\alpha([X,Y]) = \nabla_X\eta_\alpha(Y) - \\ - \nabla_Y\eta_\alpha(X) - \eta_\alpha([X,Y]) = X\eta_\alpha(Y) - Y\eta_\alpha(X) - \eta_\alpha([X,Y]) = \\ = d\eta_\alpha(X,Y)$$

$$(4)' \quad R(X,Y)\xi_\alpha = \nabla_X\nabla_Y\xi_\alpha - \nabla_Y\nabla_X\xi_\alpha - \nabla_{[X,Y]}\xi_\alpha = 0$$

$$(5)' \quad (\eta_\alpha \circ R(X,Y))Z = \eta_\alpha(\nabla_X\nabla_Y Z) - \eta_\alpha(\nabla_Y\nabla_X Z) - \eta_\alpha(\nabla_{[X,Y]} Z)$$

$$= \nabla_X\nabla_Y\eta_\alpha(Z) - \nabla_Y\nabla_X\eta_\alpha(Z) - \nabla_{[X,Y]}\eta_\alpha(Z)$$

$$= X(Y\eta_\alpha(Z)) - Y(X\eta_\alpha(Z)) - [X,Y]\eta_\alpha(Z) = 0$$

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REZIME

KONEKSIJE f-MNOGOSTRUKOSTI

U ovom radu se razmatraju afine koneksije na mnogostrukostima sa f-strukturom, kao i na mnogostrukostima sa globalno generisanom f-strukturom. Prva teorema daje potreban i dovoljan uslov da afina koneksija na mnogostrukosti sa f-strukturom bude f-koneksija (tj. u odnosu na koju je tenzorsko polje f paralelno). Druga teorema daje potreban i dovoljan uslov da afina koneksija na mnogostrukosti sa globalno generisanom f-strukturom bude $(f,\xi_\alpha,\eta_\alpha)$-koneksija (tj. koneksija u odnosu na koju su,

pored tenzorskog polja f, paralelna i vektorska polja ξ_α, i polja 1-formi η_α). U trećoj teoremi se pokazuje da svaka mnogostrukost sa globalnom f-strukturom dopušta $(f,\xi_\alpha,\eta_\alpha)$-koneksiju sa tenzorom torzije koji zavisi od same strukture. U četvrtoj teoremi se daje potreban i dovoljan uslov da mnogostrukost sa globalno generisanom f-strukturom dopušta simetričnu $(f,\xi_\alpha,\eta_\alpha)$-koneksiju. U petoj teoremi se daju neke osobine tenzora torzije i tenzora krivine neke f-koneksije na mnogostrukosti, odnosno $(f,\xi_\alpha,\eta_\alpha)$-koneksije na mnogostrukosti sa globalno generisanom f-strukturom.

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ON THE ORTHOGONAL SPACES OF THE SUBSPACES OF A RIEMANN - OTSUKI SPACE

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INTRODUCTION

In this paper we suppose that a Riemann-Otsuki space $R\text{-}O_n$ (see |1|) with the symmetric metrix tensor satisfying the relation

$$(1) \qquad Dg_{ij} = 0 \quad (\det(g_{ij}) \neq 0)$$

and the P^i_j $(\det(P^i_j) \neq 0)$ as basic objects is given. By our investigation g_{ij} and P_j satisfying throughout the relation $P^i_j g_{is} = P^i_s g_{ij}$.
The Otsuki's covariant differential of a tensor T^i_j is defined by

$$(2) \quad DT^i_j := \nabla_k T^i_j dx^k := P^i_a P^b_j (\partial_k T^a_b + {}'\Gamma_s{}^a{}_k T^s_b - {}''\Gamma_b{}^s{}_k T^a_s) dx^k$$

where the coefficients of the connections $'\Gamma$ and $''\Gamma$ and the tensor P^i_j satisfy the Otsuki's relation (see |2| (3.13))

$$(3) \qquad \partial_k P^s_r + P^i_r {}''\Gamma_i{}^s{}_r - {}'\Gamma_r{}^i{}_k P^s_i = 0.$$

Let as usual by $x^i = x^i(u^1, \dots, u^m)$ $(m < n)$ be defined an m-dimensional subspace S_m. We suppose that the rank $\|\partial x^i/\partial n^\alpha\| = m$/* and use the notation $\xi^i_\alpha := \partial x^i/\partial u^\alpha$. Let the metric tensor

/* In this paper Latin indices run from 1 to n, Greek indices $\alpha, \beta, \dots, \lambda$ run from 1 to m, but $\mu, \nu, \dots, \omega$ run from (m+1) to n. In the following the index runing from (m+1) to n will be co-or contravariant so as the original index was.

of S_m be the projection of g_{ij}, that is

(4) $$G_{\alpha\beta} := g_{ij}\xi^i_\alpha \xi^j_\beta \ .$$

We define, us usual, the contravariant components of $G_{\alpha\beta}$ by $G^{\alpha\beta}$ i.e. $G_{\alpha\beta}G^{\beta\gamma} = \delta^\gamma_\alpha$ and the contravariant components of the projection vectors by

(5) $$\xi^\alpha_a := g_{ab}G^{\alpha\beta}\xi^b_\beta \ .$$

Obviously we have $G^{\alpha\beta}=g^{ab}\xi^\alpha_a\xi^\beta_b$, where we use for the tangent and normal vectors the relations

(6) $$\xi^i_{\alpha}\underset{\mu}{N}_i = 0; \quad \underset{\mu}{N}_i\underset{\nu}{N}^i = \delta^\mu_\nu \ ; \quad \underset{\mu}{N}_j = g_{ij}\underset{\mu}{N}^i \ ; \quad \xi^i_\alpha\xi^\alpha_j + \underset{\mu}{N}^i\underset{\mu}{N}_j = \delta^i_j$$

where $\underset{\mu}{N}_i$ are mutually orthogonal unit vectors (see |2| and |3|). This relations are very useful.

Let the P-tensor of S_m be the projection rensor P^α_β defined by

(7) $$P^\alpha_\beta := P^i_j\xi^j_\beta\xi^\alpha_i \ .$$

The covariant differential over S_m of a tensor T^α_β of S_m we define by

(8) $$\overset{*}{D}T^\alpha_\beta := \overset{*}{\nabla}_\gamma T^\alpha_\beta du^\gamma = P^\alpha_\varepsilon P^\eta_\beta(\partial_\gamma T^\varepsilon_\eta + {}'\overset{*}{\Gamma}{}^\varepsilon_{\zeta\gamma}T^\zeta_\eta - {}''\overset{*}{\Gamma}{}^\zeta_{\eta\gamma}T^\varepsilon_\zeta)du^\gamma \ .$$

In |1| it was proved that the relation

(9) $${}''\overset{*}{\Gamma}{}^\alpha_{\beta\gamma} := {}''\Gamma^\alpha_{\beta\gamma} + \xi^i_{\beta\gamma}\xi^\alpha_i \quad ({}''\Gamma^\alpha_{\beta\gamma} := {}''\Gamma^i_{jk}\xi^\alpha_i\xi^j_\beta\xi^k_\gamma; \quad \xi^i_{\beta\gamma} := \frac{\partial}{\partial u^\gamma}\xi^i_\beta \ )$$

is necessary and suficient condition to be $\overset{*}{D}G_{\alpha\beta} = 0$ and it is easy to prove that the generalization

(10) $$\overset{*}{D}T_{\alpha_1\ldots\alpha_k} = \xi^{i_1}_{\alpha_1}\ldots\xi^{i_k}_{\alpha_k}\, DT_{i_1\ldots i_k}$$

of (22) in |1| holds if $T_{i_1\ldots i_k} = \xi^{\alpha_1}_{i_1}\ldots\xi^{\alpha_k}_{i_k}\,T_{\alpha_1\ldots\alpha_k}$, i.e. it is a tensor of T_m.

In Paragraph 1 we determine a sufficient condition, that a subspace S_m of a Riemann-Otsuki space will be an R-O_m

and consider some consequences of this condition. In Paragraph 2 we consider the spaces S_{n-m}, orthogonal to S_m, and determine the coefficients of their connection. At the end we prove that the condition considered in Paragraph 1 is sufficient for S_{n-m} to be a Riemann-Otsuki space.

1. THE BASIC CONDITIONS

In |1| it was proved, thet if the covariant differential of T^α , which is an element of T_m is defined by $\overset{*}{D}T^\alpha := \xi^\alpha_i DT^i$, then it must be

$$(1.1) \qquad {}'\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma := {}'\Gamma_\beta{}^\alpha{}_\gamma + \xi^i_{\beta\gamma}\xi^\alpha_i \quad ({}'\Gamma^\alpha_{\beta\gamma} := {}'\Gamma_j{}^i{}_k\,\xi^\alpha_i\xi^j_\beta\xi^k_\gamma).$$

Since we observe an Otsuki space, the connection-coefficients ${}'\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and ${}''\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and the tensor P^α_β must satisfy the relation analogous to (3), i.e. both sides of (27) |1| must vanish. This relation we can write in the form

$$(1.2) \qquad P^i_r\xi^\alpha_i \underset{\mu}{N}{}^r({}'\Gamma_\beta{}^\alpha{}_\gamma + \xi^a_{\beta\gamma})\underset{\mu}{N}{}_a - P^i_b\xi^b_\beta \underset{\mu}{N}{}^i({}''\Gamma_\beta{}^\alpha{}_\gamma + \xi^\alpha_{r\gamma})\underset{\mu}{N}{}^r = 0 .$$

Using the relations (5) and $P^i_j g_{is} = P^i_s g_{ij}$ we get

$$(1.3) \qquad P^i_r\,\xi^\alpha_i\,\underset{\mu}{N}{}^r = G^{\alpha\varepsilon}P^i_b\xi^b_\varepsilon \underset{\mu}{N}{}_i$$

or in the projection notation $P^\alpha_\mu = G^{\alpha\varepsilon}P^\mu_\varepsilon$. Substituting (1.3) in (1.2) we get the condition

$$P^i_b\xi^b_\varepsilon \underset{\mu}{N}{}_i\,|\,G^{\alpha\varepsilon}({}'\Gamma_\beta{}^a{}_\gamma + \xi^a_{\beta\gamma})\underset{\mu}{N}{}_a - \delta^\varepsilon_\beta({}''\Gamma^\alpha_{r\gamma} + \xi^\alpha_{r\gamma})\underset{\mu}{N}{}^r| = 0.$$

In the following we suppose that

$$(1.4) \qquad P^i_b\xi^b_\varepsilon \underset{\mu}{N}{}^i = 0.$$

From (1.3) it follows that in this case $P^\mu_\alpha = P^\alpha_\mu = 0$, and it is a sufficient condition for S_m that Otsuki's relation between coefficients of connections ${}'\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$ and ${}''\overset{*}{\Gamma}{}_\beta{}^\alpha{}_\gamma$. and the tensor P^α_β could be satisfied.

Now, we prove some consequences of (1.4). It is known that in Otsuki spaces there exists a tensor Q^i_j satisfying the relation

$$P^i_j\, Q^j_s = \delta^i_s \,. \tag{1.5}$$

Let the projection of P^i_j in the direction of the vectors orthogonal to S_m be

$$P^\mu_\nu := P^i_j \underset{\mu}{N}_i \underset{\nu}{N}^j \tag{1.6}$$

Now, we suppose that there are tensors $\overset{*}{Q}{}^\alpha_\beta$ and $\tilde{Q}^\mu_\nu$ so that

$$P^\alpha_\beta \overset{*}{Q}{}^\beta_\gamma = \delta^\alpha_\gamma \quad ; \quad P^\mu_\nu \tilde{Q}^\nu_\sigma = \delta^\mu_\sigma \tag{1.7}$$

hold.

THEOREM 1. *From (1.4) it follows that* $\overset{*}{Q}{}^\alpha_\beta = Q^i_j \xi^\alpha_i \xi^j_\beta = Q^\alpha_\beta$ *and* $\tilde{Q}^\mu_\nu = Q^i_j \underset{\mu}{N}_i \underset{\nu}{N}^j = Q^\mu_\nu$.

P r o o f. Substituting (7) in (1.7), multiplying it by ξ^k_α, using the last relation of (6) and (1.4) we get

$$P^k_j \xi^j_\beta \overset{*}{Q}{}^\beta_\varepsilon = \xi^k_\varepsilon \,.$$

Multiplication by $Q^\ell_k \xi^\alpha_\ell$ according to $\xi^i_\alpha \xi^\beta_i = \delta^\beta_\alpha$ gives the affirmation of the theorem. The second part of the theorem follows in the same way, but we must take the vectors $\underset{\mu}{N}^i$, instead of the ξ^i_α.

THEOREM 2. *From* $P^\mu_\alpha = 0$ *it follows that* $Q^\alpha_\mu = Q^\mu_\alpha = 0$.

P r o o f. According to the definition of P^μ_β and (1.4) we have $P^\mu_\beta = P^i_j \underset{\mu}{N}_i \xi^j_\beta = 0$. Multiplying it by $\xi^\beta_k Q^k_s$ using (6) and (1.5) we get

$$\underset{\mu}{N}_s = P^\mu_\varrho \underset{\rho}{N}_k Q^k_s \,.$$

Further, multiplication by Q^σ_μ and ξ^s_α gives

$$Q^k_s \xi^s_\alpha \underset{\sigma}{N}_k = Q^\sigma_\alpha = 0 \,.$$

THEOREM 3. *From the relation (1.4) it follows that*

$$(1.8)\qquad a)\quad \begin{aligned} P^i_j\xi^\alpha_i &= P^\alpha_\beta\xi^\beta_j \\ Q^i_j\xi^\alpha_i &= Q^\alpha_\beta\xi^\beta_j \end{aligned} \qquad b)\quad \begin{aligned} P^i_j N_{\mu i} &= P^\mu_\nu N_{\nu j} \\ Q^i_j N_{\mu i} &= Q^\mu_\nu N_{\nu j} \end{aligned}$$

$$(1.9)\quad M^i_j\xi^\alpha_i = M^\alpha_\beta\xi^\beta_j \qquad M^i_j N_{\mu i} = M^\mu_\nu N_{\nu j} \qquad (M^i_j = P^i_a P^a_j;\ M^\alpha_\beta = P^\alpha_\varepsilon P^\varepsilon_\beta;\ M^\mu_\nu = P^\mu_\sigma P^\sigma_\nu)$$

The multiplication of (1.4) by ξ^β_k aocording to (10) and (1.6) proves the statement of this theorem. Relations (1.8) and (1.9) are very useful and they will be often applied in the followings.

Relations (1.8) and (1.9) mean that they hold an eigen quality in all the space for the vectors ξ^α_i and $N_{\mu i}$. The subspace S_m is an eigen space, and its orthogonal space S_{n-m} is an eigen space too. If $m = n - 1$, then relation (1.8b) is a simple eigen property. From (1.4) it follows directly that

$$P^i_j = P^\alpha_\beta\xi^i_\alpha\xi^\beta_j + P^\mu_\nu N^i_\mu N_{\nu j}$$

and according to the statement of the Theorem 2 it follows that

$$Q^i_j = Q^\alpha_\beta\xi^i_\alpha\xi^\beta_j + Q^\mu_\nu N^i_\mu N_{\nu j} .$$

One of consequences of (1.4) is that the relations (24) and (26) of 1 are equivalent, i.e. (1.1) holds.

2. THE CONNECTION OF THE ORTHOGONAL SPACES

In this paragraph we extend the definition of covariant differential $\overset{*}{D}$ on the elements orthogonal to S_m, but defined over it. The coefficients of the connection of co- and contravariant part of the orthogonal space we denote by $'\Lambda^\mu_{\nu\gamma}$ and $''\Lambda^\mu_{\nu\gamma}$ respectively. We must determine the conditions that the Otsuki´s relation (3) between this coefficients of connections and the tensor P^μ_ν will be satisfied. The tensor P^μ_ν is the projection of the tensor P^i_j on the orthogonal subspace S_{n-m}. In the following we consider a covariant vector Y_i orthogonal to S_m and defined over it. It is expressible in the from

(2.1) $$Y_i = \underset{\mu}{N}_i Y_\mu .$$

Now we define the covariant differential $\overset{*}{D}Y_\mu$ as a projection of DY_i onto the (n-m)-dimensional direction orthogonal to S_m, i.e.

(2.2) $$\overset{*}{D}Y_\mu := g^{ij} \underset{\mu}{N}_j DY_i = \underset{\mu}{N}^i DY_i .$$

According to (2) and (2.1) it follows that

$$\overset{*}{D}Y_\mu = g^{ij} \underset{\mu}{N}_j P^a_i \underset{\rho}{N}_a (\partial_\gamma Y_\rho) du^\gamma - g^{ij} \underset{\mu}{N}_j P^a_i ("\Gamma^{\ s}_{a\ k} \underset{\rho}{N}_s \xi^k_\gamma - \partial_\gamma \underset{\rho}{N}_a) Y_\rho du^\gamma .$$

Using (6), (1.6) and the notations $\underset{\sigma}{N}_a "\Lambda^{\rho}_{\sigma\gamma} := "\Gamma^{s}_{a\ k} \underset{\rho}{N}_s \xi^k_\gamma - \partial_\gamma \underset{\rho}{N}_a$ or

(2.3) $$"\Lambda_{\sigma\ \gamma}^{\ \rho} := ("\Gamma^{\ s}_{a\ k} \underset{\rho}{N}_s \xi^k_\gamma - \partial_\gamma \underset{\rho}{N}_a) \underset{\sigma}{N}^a$$

we get

(2.4) $$\overset{*}{D}Y_\mu = P^\rho_\mu (dY_\rho - "\Lambda^\sigma_{\rho\gamma} Y_\sigma du^\gamma) = \overset{*}{\nabla}_\gamma Y_\mu du^\gamma .$$

In the same way we define the covariant differential $\overset{*}{D}Y^\mu$ of a contravariant vector Y^μ which has the contravariant components in the basic $R\text{-}O_n$ space. Let Y^i be orthogonal to S_m. Now we define:

(2.5) $$\overset{*}{D}Y^\mu := \underset{\mu}{N}_i DY^i .$$

Since Y^i is expressible in the from $Y^i = \underset{\mu}{N}^i Y^\mu$, using (2) it is not difficult to get, that if

$$\underset{\rho}{N}^a \,{}'\Lambda^{\ \rho}_{\nu\gamma} := {}'\Gamma^{\ a}_{s\ k} \underset{\nu}{N}^s \xi^k_\gamma + \partial_\gamma \underset{\nu}{N}^a$$

or

(2.6) $$'\Lambda_{\nu\ \gamma}^{\ \rho} := ({}'\Gamma^{\ a}_{s\ k} \underset{\nu}{N}^s \xi^k_\gamma + \partial_\gamma \underset{\nu}{N}^a) \underset{\rho}{N}_a$$

then according to (2.6) relation (2.5) has the form

(2.7) $$\overset{*}{D}Y^\mu = P^\mu_\nu (\partial_\gamma Y^\nu + {}'\Lambda_{\rho\ \gamma}^{\ \nu} Y^\rho) du^\gamma = \overset{*}{\nabla}_\gamma Y^\mu du^\gamma .$$

Relations (2.4) and (2.7) show that in the subspace S_{n-m} it is possible to define a covariant differential, like in the Otsuki´s spaces |2|.

Coefficients $'\Lambda_{\nu\ \gamma}^{\ \rho}$ and $"\Lambda_{\nu\ \gamma}^{\ \rho}$ are coefficients of connections of the space S_{n-m}. In this paper we consider an Otsuki space, and so we must determine conditions that Otsuki´s relation

$$(2.8) \qquad \partial_\gamma P^\mu_\sigma + P^\nu_\sigma {}''\Lambda_\nu{}^\mu{}_\gamma - P^\mu_\nu {}'\Lambda_\sigma{}^\nu{}_\gamma = 0$$

will be satisfied. Substituting (2.3), (2.6) and (1.6) in(2.8), using the fact that ${}'\Gamma_j{}^i{}_k$, ${}''\Gamma_j{}^i{}_k$ and P^i_j satisfy (3), we get that it must be

$$(2.9) \qquad P^i_r \xi^r_\alpha N_{\mu i}({}'\Gamma_a{}^s{}_\gamma \xi^\alpha_s - \xi^\alpha_{a\gamma}) N^a_\sigma - P^i_j \xi^\alpha_i N^j_\sigma({}''\Gamma_s{}^a{}_\gamma \xi^s_\alpha + \xi^a_{\alpha\gamma}) N_{\mu a} = 0.$$

According to the supposition (1.4), using (1.3), it follows that (2.9) is satisfied. It means, that the following holds:

THEOREM 4. *The assumption (1.4) is a sufficient condition that Otsuki's relation between coefficients of connection* ${}'\Lambda_\nu{}^\mu{}_\gamma$ *and* ${}''\Lambda_\nu{}^\mu{}_\gamma$ *and the tensor* P^μ_ν *will be satisfied, and so* S_{n-m} *is a Riemann-Otsuki space .*

After all we can define the covariant differential of a mixed tensor involving there kinds of indices, for instance a tensor $T^{i\alpha\mu}_{j\beta\nu}$. Now it is

$$(2.10) \qquad \overset{*}{D}T^{i\alpha\mu}_{j\beta\nu} := P^i_a P^b_j P^\alpha_\varepsilon P^\eta_\beta P^\mu_\rho P^\sigma_\nu (\partial_\gamma T^{a\varepsilon\rho}_{b\eta\sigma} + {}'\Gamma_s{}^a{}_\gamma T^{s\varepsilon\rho}_{b\eta\sigma} + {}'\overset{*}{\Gamma}_\chi{}^\varepsilon{}_\gamma T^{a\chi\rho}_{b\eta\sigma} +$$
$$+ {}'\Lambda_\tau{}^\rho{}_\gamma T^{a\varepsilon\tau}_{b\eta\sigma} - {}''\Gamma_b{}^s{}_\gamma T^{a\varepsilon\rho}_{s\eta\sigma} - {}''\overset{*}{\Gamma}_\mu{}^\chi{}_\gamma T^{a\varepsilon\rho}_{b\chi\sigma} - {}''\Lambda_\sigma{}^\tau{}_\gamma T^{a\varepsilon\rho}_{b\eta\tau}) du^\gamma .$$

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REZIME

O ORTOGONALNIM PROSTORIMA PODPROSTORA RIEMANN-OTSUKIJEVOG PROSTORA

U paragrafu 1 dati su dovoljni uslovi da je potprostor S_m jednog Riemann-Otsuki-evog prostora isto $R-O_m$ i date su neke posledice ovog uslova. U paragrafu 2 uočeni su podprostori S_{n-m} ortogonalni na S_m, i odredjeni koeficijenti njihove koneksije i dokazano je da su uslovi iz paragrafa 1 dovoljni da bi S_{n-m} bio jedan Riemann-Otsuki-ev prostor.

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ON A STRUCTURE ϕ SATISFYING $(\phi^2+1)(\phi^2-a)=0$

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1. In |1| and |2| a unified notation is given of almost complex structure and almost contact structure by the introduction of a tensor field of type (1,1) on M^n such that $f^3+f=0$ and the rank $f=k$ is a constant everywhere The necessary and sufficient condition for an n-dimensional manifold to admit a tensor field $f\neq 0$ of type (1,1) such that $f^3+f=0$ is that k=2m and that the group of the tangent bundle of the manifold can be reduced to the group $U_{(m)}\times O_{(n-2m)}$. In |3| and |4| the structure $\phi^4\pm\phi^2=0$ is studied and the necessary and sufficient condition is given when M^n admits such a structure. In this paper, we want to introduce a tensor field of type (1,1) which satisfies the condition $(\phi^2+1)(\phi^2-a)=0$.

2. Let M^n be an n-dimensional differentiable manifold of the class C^∞ and let ϕ be a tensor field, $\phi^2\neq a$, $\phi^2\neq -1$ of type (1,1) and of class C^∞ such that n=2m,

$$(2.1)\qquad (\phi^2+1)(\phi^2-a)=0\ ,\quad a\in R^+\ ,\ a\neq 1$$

and rank $\phi=\frac{1}{2}$ (rank $\phi^2+\dim M^n$)=r=const.

For a differentiable manifold with a structure which satisfies such conditions we say that it asmits an $\phi(+1,a)$ structure.

Let

$$(2.2)\qquad \ell=\frac{\phi^2-a}{-1-a}\ ,\qquad m=\frac{\phi^2+1}{a+1}\ .$$

Then ℓ and m are complementary projection operators since $m^2 = m$, $\ell^2 = \ell$, $\ell m = m\ell = 0$, $\ell + m = 1$, which can easily be verified.

THEOREM 2.1. *Let ϕ satisfy conditions (2.1) and let ℓ and m be defined by condition (2.2). We then have*

$$(2.3) \qquad \phi\ell = \frac{\phi^3 - \phi a}{-1-a}, \qquad \phi m = \frac{\phi^3 + \phi}{a+1}$$

$$\phi^2\ell = \frac{-\phi^2 + a\phi^2 + a - a\phi^2}{-1-a} = \ell, \quad \phi^2 m = \frac{\phi^4 + \phi^2}{a+1} = \frac{\phi^2 + a\phi^2 + a - \phi^2}{a+1} = am.$$

Let L and M be complementary distributions which correspond to projections ℓ and m respectively. Then, (because of (2.3)), ϕ acts on L as an almost complex structure and on M as a structure for wich ϕ^2 is a homothety with coefficient of the homothety a. If ϕ is of constant rank r, then the dimensions of L and M are $2r-n$ and $2n-2r$ respectively. Obviously we have $n \leq 2r \leq 2n$.

REMARK 2.1. If the rank of ϕ is n, then $\phi^2 + 1 = 0$. Consequently, the $\phi(+1,a)$ structure of maximal rank is an almost complex structure.

REMARK 2.2. If the rank of ϕ is $n/2$, then $\phi^2 - a = 0$. Hence the $\phi(+1,a)$ structure of minimal rank is a structure for which ϕ^2 is a homothety with coefficient of the homothety a.

We shall now examine under which conditions the differentiable manifold admits a $\phi(+1,a)$ structure.

3. We now introduce a local coordinate system in the manifold and denote by ϕ_i^j, ℓ_i^j, m_i^j the local components of the tensors ϕ, ℓ, m respectively. We also introduce a positive definite Riemannian metric in the manifold and take $2r-n$ mutually orthogonal unit vectors v_a^i $(a,b,c,\ldots=1,2,\ldots,2r-n)$ in L and $2(n-r)$ to be mutually orthogonal unit vectors v_A^j $(A,B,C,\ldots =2r-n+1,\ldots,n)$ in M. We then have

$$(3.1) \qquad \begin{aligned} \ell_i^j\, v_b^i &= v_b^j, & \ell_i^j\, v_B^i &= 0, \\ m_i^j\, v_b^i &= 0, & m_i^j\, v_B^i &= v_B^j. \end{aligned}$$

If we denote by (s_i^a, s_i^A) the matrix inverse to (v_b^j, v_B^j) then s_i^a and s_i^A are both components of linearly indenpendent covariant vectors and satisfy the relations:

$$(3.2) \qquad s_i^a\, v_b^i = \delta_b^a, \quad s_i^a\, v_B^i = 0, \quad s_i^A\, v_b^i = 0, \quad s_i^A\, v_B^i = \delta_B^A,$$

$$(3.3) \qquad s_i^a\, v_a^j + s_i^A\, v_A^j = \delta_i^j.$$

If we put

$$(3.4) \qquad p_{ki} = s_k^a\, s_i^a + s_k^A\, s_i^A$$

then p_{ki} is a globally well-defined positive definite Riemannian metric with respect to which (v_b^j, v_B^j) form an orthogonal frame such that

$$s_k^a = p_{ki}\, v_a^i, \quad s_k^A = p_{ki}\, v_A^i.$$

Now from (3.1) and (3.2) we find that

$$(\ell_i^j\, s_j^a) v_b^i = \delta_b^a, \qquad (\ell_i^j\, s_i^a) v_B^i = 0,$$

$$(m_i^j\, s_j^A) v_b^i = 0, \qquad (m_i^j\, s_j^A) v_B^i = \delta_B^A,$$

which show that:

$$(3.5) \qquad \begin{aligned} \ell_i^j\, s_j^a &= s_i^a, & m_i^j\, s_j^A &= s_i^A, \\ m_i^j\, s_j^a &= 0, & \ell_i^j\, s_j^A &= 0. \end{aligned}$$

On the other hand, from $\ell_i^j\, v_a^i = v_a^j$ we find that

$$\ell_k^j\, s_i^A\, v_a^k = s_i^a\, v_a^j, \qquad \ell_k^j(\delta_i^k - s_i^A\, v_A^k) = s_i^a\, v_a^j$$

that is,

$$(3.6) \qquad \ell_i^j = s_i^a\, v_a^j.$$

Similarly we get

$$(3.7) \qquad m_i^j = s_i^B \, v_B^i \, .$$

If we put

$$(3.8) \qquad \ell_{ki} = \ell_k^r \, p_{ri} \, , \quad m_{ki} = m_k^r \, p_{ri}^r \, ,$$

we find from (3.6), (3.7) and (3.4)

$$(3.9) \qquad \ell_{ki} = s_k^a \, s_i^a \, , \qquad m_{ki} = s_k^A \, s_i^A \, ,$$

$$(3.10) \qquad \ell_{ki} = \ell_{ik} \, , \qquad m_{ki} = m_{ik} \, , \quad \ell_{ik} + m_{ik} = p_{ik} \, .$$

We can also easily verify the following relations

$$(3.11) \qquad \ell_k^r \, \ell_i^q \, p_{rq} = \ell_{ki} \, , \quad \ell_k^r \, m_i^q \, p_{rq} = 0 \, ,$$

$$m_k^r \, m_i^q \, p_{rq} = m_{ki} \, .$$

For any two vectors x,y with components x^i, y^i let us put

$$(3.12) \qquad m^*(x,y) = m_{rq} x^r y^q \, , \; p(x,y) = p_{rq} x^r y^q \, ,$$

$$(3.13) \qquad \bar{g}(x,y) = \tfrac{1}{2}(p(x,y) + p(\phi x, \phi y) + m^*(x,y)) \, .$$

Then we have

$$m^*(v_A, v_a) = p(v_A, v_a) = 0,$$

$$\bar{g}(v_A, v_a) = \tfrac{1}{2}(p(v_A, v_a) + p(\phi v_A, \phi v_a) + m^*(v_A, v_a)) = 0 \, .$$

Thus L and M are orthogonal with respect to $\bar{g}$. Furthermore, it is easy to verify by using (3.8) and (3.10) that

$$p(\phi v_a, \phi v_b) = \ell_{rq} \phi_h^r \, \phi_j^q \, v_a^h \, v_b^j \, ,$$

$$p(\phi v_a, \phi v_b) + m^*(\phi v_a, \phi v_b) = p_{rq} \, \phi_h^r \, \phi_j^q \, v_a^h \, v_b^j \, ,$$

$$p(\phi^2 v_a, \phi^2 v_b) = p_{rq} \, v_a^r \, v_b^q \, .$$

These relations lead to the following:

$$\bar{g}(\phi x,\phi y) = \bar{g}(x,y) \quad \text{for all } x,y \text{ in } L. \tag{3.14}$$

Let M_1 be a space such that for $x \in M_1$, $\phi(x) = \sqrt{a}x$ and let M_2 be the distribution orthogonal to M_1 in M with respect to $\bar{g}$. We choose an orthonormal basis $u_{n-r+1},\dots,u_{2(n-r)}$ with respect to $\bar{g}$ for M_2. Furthermore, let $e_1,\dots,e_{2r-n}$ be an orthonormal basis for L with respect to $\bar{g}$. Using $\bar{g}$ we can defined a Riemannian metric g on M^n by

$$g(e_i,e_k) = \bar{g}(e_i,e_k),\quad g(e_i,u_\alpha) = \bar{g}(e_i,u_\alpha),\ g(u_\alpha,u_\beta) = \bar{g}(u_\alpha,u_\beta)$$

$$g(e_i,\phi(u_\alpha)) = \bar{g}\ (e_i,\phi(u_\alpha)),\quad g(\phi(u_\alpha),u_\beta) = 0,\ g(\phi(u_\alpha),\phi(u_\beta))=$$
$$= \delta_{\alpha\beta},\quad 1 \leq i,\ k \leq 2r-n,\quad n-r+1 \leq \alpha,\beta \leq 2n-r\ .$$

Then g is well-defined because if $\bar{u}_{n-r+1},\dots,\bar{u}_{2(n-r)}$ is another orthonormal basis for M_2 then for $\bar{u}_\alpha = Z^\beta_\alpha u_\beta$ we have

$$\delta_{\alpha\gamma} = \bar{g}(\bar{u}_\alpha,\bar{u}_\gamma) = \bar{g}(z^\beta_\alpha u_\beta,\ z^\varepsilon_\gamma u_\varepsilon) = z^\beta_\alpha z^\varepsilon_\gamma \delta_{\beta\varepsilon} = z^\beta_\alpha z^\beta_\gamma$$

and

$$g(\phi(\bar{u}_\alpha),\phi(\bar{u}_\gamma)) = g(z^\beta_\alpha\ \phi(u_\beta),z^\varepsilon_\gamma\ \phi(u_\varepsilon)) = z^\beta_\alpha z^\varepsilon_\gamma\ g(\phi(u_\beta),\phi(u_\varepsilon)) =$$
$$= z^\beta_\alpha\ z^\beta_\gamma = \delta_{\alpha\gamma}.$$

This means that there is a Riemannian metric g with respect to which L, M_1, M_2 are mutually orthogonal and

$$g(\phi x,\phi y) = g(x,y) \quad \text{for all } x,y \text{ in } L.$$
$$g(\phi x,\phi y) = a\cdot g(x,y) \quad \text{for all } x,y \text{ in } M.$$

THEOREM 3.1. *If in an* n-*dimension manifold* M^n (n=2m) *a* $\phi(+1,a)$ *structure of rank* r *is given, then there exist complementary distributions* L *of dimension* 2r-n, *and* M *of dimension* 2(n-r) *and a positive definite Riemannian metric* g *with respect to which* L *and* M *are orthogonal, and, furthermore, such that*

$$g(x,y) = g(\phi x,\phi y) \qquad x,y \in L$$
$$ag(x,y) = g(\phi x,\phi y) \qquad x,y \in M.$$

4. Take a vector e in the distribution L. Then the vector $\phi(e)$ is also in L and perpedikular to e, and moreover has the same lenght as e with respect to the metric g. Consequently we can choose $2r-n=2(r-m)$ orthonormal vectors in L such that

$$\phi(e_1) = e_{r-m+1}, \phi(e_2) = e_{r-m+2}, \dots, \phi(e_{r-m}) = e_{2(r-m)},$$

and in M an orthonormal basis $e_{2(r-m)+1}, \dots, e_n$ such that $e_{r+1}, \dots, e_n$ are in M_2 and that $\phi(e_{r+1}) = -\sqrt{a}e_{2(r-m)+1}, \dots, \phi(e_n) = -\sqrt{a}e_r$.

Then with respect to this orthonormal frame $\{e_1, \dots, e_n\}$ the tensors g_{ji} and ϕ_j^i have components

$$(4.1)\quad g = \begin{bmatrix} E_{r-\frac{n}{2}} & 0 & 0 & 0 \\ 0 & E_{r-\frac{n}{2}} & 0 & 0 \\ 0 & 0 & E_{n-r} & 0 \\ 0 & 0 & 0 & E_{n-r} \end{bmatrix} \phi = \begin{bmatrix} 0 & E_{r-\frac{n}{2}} & 0 & 0 \\ -E_{r-\frac{n}{2}} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{a}E_{n-r} & 0 \\ 0 & 0 & 0 & -\sqrt{a}E_{n-r} \end{bmatrix}$$

Such a frame is an adapted frame of the $\phi(+1,a)$ structure. Let $\{\bar{e}_i\}$ be another adapted frame in wich g and ϕ have the same components as (4.1).

Put $\bar{e}_i = \gamma_i^j e_j$. Then γ has a matrix of the form

$$\gamma = \begin{bmatrix} A_{r-\frac{n}{2}} & B_{r-\frac{n}{2}} & 0 & 0 \\ -B_{r-\frac{n}{2}} & A_{r-\frac{n}{2}} & 0 & 0 \\ 0 & 0 & C_{n-r} & 0 \\ 0 & 0 & 0 & D_{n-r} \end{bmatrix}.$$

This means that the group of the tangent bundle of the manifold can be reduced to $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$. Conversely, if the group of the tangent bundle of the manifold can be reduced to $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$ then we can define a positive definite Riemannian metric g and a $\phi(+1,a)$ structure with matrices

(4.1) with respect to the adapted frames. Then we have

$$\phi^2 = \begin{bmatrix} -E_{r-\frac{n}{2}} & 0 & 0 & 0 \\ 0 & -E_{r-\frac{n}{2}} & 0 & 0 \\ 0 & 0 & aE_{n-1} & 0 \\ 0 & 0 & 0 & aE_{n-r} \end{bmatrix}, \phi^4 = \begin{bmatrix} E_{r-\frac{n}{2}} & 0 & 0 & 0 \\ 0 & E_{r-\frac{n}{2}} & 0 & 0 \\ 0 & 0 & a^2E_{n-r} & 0 \\ 0 & 0 & 0 & a^2E_{n-r} \end{bmatrix}$$

and it is easily verified that $(\phi^2+1)(\phi^2-a)=0$. From this we have

THEOREM 4.1. *A necessary and sufficient condition for the* n-*dimensional manifold to admit a* $\phi(+1,a)$ *structure is that the group of the tangent bundle can be reduces to the group* $U_{r-\frac{n}{2}} \times O_{n-r} \times O_{n-r}$.

It is known from |3| that if the structural group of a manifold M^n is reduced to $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$, then M^n admits a $\phi(4,+2)$ structure. L is the subspace of M^n on which $\phi^2=-1$, while the complement M of the space L in M^n admits an almost tangent structure.

From this paper it follows that if the structural group of manifold M^n is reduced to $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$, then M^n admits a $\phi(+1,a)$ structure. L is the subspace of M^n on which $\phi^2=-1$, while the complement M of space L in M^n admits the structure ϕ_M, for which $\phi_M^2=a$.

From this we have:

THEOREM 4.2. *If the structural group of manifold* M^n *is reduced to* $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$, *then* M^n *admits a* $\phi(4,+2)$ *and* $\phi(+1,a)$ *structure.* L *is the subspace of* M^n *on which* $\phi^2=-1$ *for both structures. Complement* M *of the space* L *in* M^n *admits structure for which* $\phi_M^2=0$ *or* $\phi_M^2=a$.

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REZIME

O STRUKTURI ϕ KOJA ISPUNJAVA USLOV $(\phi^2+1)(\phi^2-a)=0$

Potreban i dovoljan uslov da se n-dimenzionalna mnogostrukost može snabdeti $\phi(+1,a)$ strukturom je da se grupa tangentnog bandla može reducirati do $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$. Ako se strukturna grupa može reducirati do $U_{(r-\frac{n}{2})} \times O_{(n-r)} \times O_{(n-r)}$ tada se mnogostrukost može snabdeti i $\phi(4,+2)$ i $\phi(+1,a)$ strukturom. U prvom slučaju je $\phi_L^2=-1$, a na komplementarnom prostoru M je $\phi_M^2=0$. U drugom slučaju $\phi_L^2=-1$, a na M je $\phi_M^2=a$.

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SUBALGEBRAS OF COMMUTATIVE SEMIGROUP SATISFYING THE LAW $x^r = x^{r+m}$

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ABSTRACT

An algebra with a type Ω and a carrier A is an Ω-subalgebra of a semigroup S if $A \subseteq S$ and if there is a mapping $\omega \mapsto \bar{\omega}$ of Ω into S such that $\omega(a_1,\ldots,a_n) = \bar{\omega}a_1\ldots a_n$, for every n-ary operator $\omega \in \Omega$ and the sequence of elements $a_1,\ldots,a_n$ of A. If $\underline{C}$ is a class of semigroups then by $\underline{C}(\Omega)$ is denoted the class of Ω-algebras (i.e. algebras of the type Ω) which are subalgebras of semigroups belonging to $\underline{C}$. It is well known (see |1| p. 185 or |4| p. 78) that SEM(Ω) is the class of all Ω-algebras. It is also known (|5|) that ABSEM(Ω) is a variety. The object of our investigations is the set V of varietis $\underline{V}$ of semigroups such that $\underline{V}(\Omega)$ is also a variety. In Theorem 1. of this paper we show that $\underline{C}_{r,m}(\Omega)$ is a variety only if $r=1$ or Ω does not contain n-ary operators for $n \geq 2$, where $\underline{C}_{r,m}$ is the class of commutative semigroups which satisfy the law $x^r = x^{r+m}$.

0. MAIN RESULTS

First, we note that if Ω is a set of finitary operators then $\Omega(n) = \{\omega \in \Omega \mid \omega$ is an n-ary operator$\}$. Obviously an Ω-algebra is an Ω-subalgebra of a semigroup S iff the corresponding restriction $\Omega \setminus \Omega(0)$-algebra is an $\Omega \setminus \Omega(0)$-subalgebra of S. Thus, we can assume that $\Omega(0) = \emptyset$ i.e. that Ω does not contain nullary operators.

THEOREM 1. $\underline{C}_{r,m}(\Omega)$ *is a variety iff* r=1 *or* $\Omega = \Omega(1)$.

THEOREM 2. *Let* A *be a nonempty set,* r *and* m *two positive integers, and* L *a subsemigroup of the semigroup* T_A *of all transformations of* A, *such that* $L \in \underline{C}_{r,m}$. *Then, there exists a semigroup* $M \in \underline{C}_{r,m}$ *with the following properties:*

(i) L *is a subsemigroup of* M;

(ii) $A \subseteq M$;

(iii) $(\forall a \in A,\ \exists \phi \in L)\phi(a) = \phi a$. (ϕa *is the "product" of* ϕ *and* a *in* M)

Before giving the formulation of the last theorem, we have to give some preliminary definitions. Namely, if A is a nonempty set, then by O(A) is denoted the set of finitary (not nullary) operations on A, i.e. $O(A) = \bigcup_{n=1}^{\infty} O_n(A)$, where $O_n(A) = A^{A^n}$ consists of all n-ary operations on A. If $L \subseteq O(A)$, then $L(n) = L \cap O_n(A)$. An infinite collection $\{\overset{i}{+} \mid i=1,2,\ldots\}$ of partial binary operations can be defined on O(A) by

(1) $\phi \in O_n(A),\ \psi \in O_m(A),\ i \leq n \ \Rightarrow$

$$\phi \overset{i}{+} \psi(x_1,\ldots,x_{m+n-1}) = \phi(x_1,\ldots,x_{i-1},\psi(x_i,\ldots,x_{i+m-1}),\ ,x_{i+m},\ldots,x_{m+n-1})$$

(See for example |6| p. 7-49 or |3| p. 9). We have that $(O(A),\overset{i}{+})$ is a monoid. Further on, for the operation $\overset{i}{+}$ a usual multiplicative notation will be used. An operation $\phi \in O_n(A)$ is called commutative if

(2) $$\phi(a_1,\ldots,a_n) = \phi(a_{i_1},\ldots,a_{i_n})$$

for every sequence $a_1,\ldots,a_n \in A$ and permutation $\nu \mapsto i_\nu$ of $N_n = \{1,2,\ldots,n\}$.

THEOREM 3. *Let* L *be a commutative subsemigroup of the semigroup* O(A) *such that all the operations belonging to* L *are commutative and* $\phi \overset{i}{+} \psi = \phi\psi$, *for any* $\phi,\psi \in L$ *and* $i \in \{1,\ldots,n\}$ *where* $\phi \in L(n)$. *Let* m *be a positive integer and assume that* L *satisfies the following statement:*

(*) If $\phi_1,\ldots,\phi_p \in L$, $\phi_\nu \in L(n_\nu+1)$ *and* $i_1,\ldots,i_p,j_1,\ldots,j_p,\alpha_1,\ldots$ $\ldots,\alpha_q,\beta_1,\ldots,\beta_q$ *are positive integers such that:*

(3) $i_\nu \equiv j_\nu (\mathrm{mod}\, m), \quad \alpha_\lambda \equiv \beta_\lambda (\mathrm{mod}\, m)$

and

(4)
$$1 + i_1 n_1 + \ldots + i_p n_p = \alpha_1 + \ldots + \alpha_q$$
$$1 + j_1 n_1 + \ldots + j_p n_p = \beta_1 + \ldots + \beta_q ,$$

then

(5)
$$\phi_1^{i_1}\ldots\phi_p^{i_p}(x_1^{\alpha_1},\ldots,x_q^{\alpha_q}) = \phi_1^{j_1}\ldots\phi_p^{j_p}(x_1^{\beta_1},\ldots,x_q^{\beta_q})$$

is an identity equation on A. *Then, there exists a semigroup* $M \in \underline{C}_{1,m}$ *and a homomorphism* $\phi \mapsto \bar{\phi}$ *from* L *into* M *such that the following statements are satisfied:*

(i) $(\forall \phi \in L(1))\ \bar{\phi} = \phi;$

(ii) $A \subseteq M$;

(iii) $(\forall a_1,\ldots,a_n \in A,\ \phi \in L(n))\phi(a_1,\ldots,a_n) = \bar{\phi}a_1\ldots a_n$.

Obviously, the part of Theorem 1. for r=1, is a special case of Theorem 2.. But the corresponding generalization for $r \geq 2$ is not true, for, by Theorem 1., $\underline{C}_{r,m}(\Omega)$ is not a variety if $r \geq 2$ and $\Omega \neq \Omega(1)$. It should also be noticed that if $L \neq L(1)$ then the homomorphism $\phi \mapsto \bar{\phi}$ is not a monomorphism, for $\bar{\phi} = \overline{\phi^{m+1}}$ but if $\phi \in L(n)$ $n \geq 2$, then $\phi^{m+1} \neq \phi$. This suggests the problem of finding the set of varieties $\underline{V}$ of semigroups such that every subsemigroup $L \in \underline{V}$ of $O(A)$ (or more special of $T_A = O_1(A)$) can be embedded in a semigroup $M \in \underline{V}$.

1. P r o o f of Theorem 1. Identities in $\underline{C}_{r,m}(\Omega)$. Obviously, if A is an Ω-algebra belonging to $\underline{C}_{r,m}(\Omega)$ then A satisfies the following identity equations:

(**) $\phi(x_1,\ldots,x_n) = \phi(x_{i_1},\ldots,x_{i_n})$

for every $\phi \in \Omega(n)$ and permutation $\nu \mapsto i_\nu$ of N_n, i.e. the all operations of the algebra are commutative,

(***) $\phi\psi = \psi\phi = \phi \overset{i}{+} \psi$

for any $\phi, \psi \in \Omega$ and $i \in \{1,2.,,,.n\}$ where $\phi \in \Omega(n)$;

and (*′) which is obtained from (*) in Theorem 3., replacing L with Ω and (3) with

$$(3') \qquad i_\nu = j_\nu \ \text{ or } \ (i_\nu, j_\nu \geq r \ \text{ and } \ i_\nu \equiv j_\nu (\mathrm{mod}\, m))$$
$$\alpha_\lambda = \beta_\lambda \ \text{ or } \ (\alpha_\lambda, \beta_\lambda \geq r \ \text{ and } \ \alpha_\lambda \equiv \beta_\lambda (\mathrm{mod}\, m))$$

It can be easily seen that all identity equations, which hold in all Ω-algebras belonging to $\underline{C}_{r,m}(\Omega)$, are consequences of (*′), (**) and (***). Namely, let ξ be an Ω-term (a term with operational symbols from Ω) with i_ν occurrences of the operator ω_ν, and α_λ occurrences of the variable x_λ. Then, by a finite number of applications of (**) and (***) we can obtain that

$$\xi = \omega_1^{i_1} \ldots \omega_p^{i_p}(x_1^{\alpha_1}, \ldots, x_q^{\alpha_q})$$

is an identity in $\underline{C}_{r,m}(\Omega)$. We have to show that if (3′) is not satisfied then (5) is not an identity in $\underline{C}_{r,m}(\Omega)$. Let F be a semigroup in $\underline{C}_{r,m}$ which is freely generated by $\Omega \cup \{e_1, e_2, \ldots \ldots, e_k, \ldots\}$, where $e_\nu \notin \Omega$. By putting

$$\omega(u_1, \ldots, u_n) = \omega u_1 \ldots u_n ,$$

for every $\omega \in \Omega(n)$ and $u_1, \ldots, u_n \in F$ we obtain an Ω-algebra F, which, obviously, belongs to $\underline{C}_{r,m}(\Omega)$. If (3′) is not satisfied, then

$$\omega_1^{i_1} \ldots \omega_p^{i_p} e_1^{\alpha_1} \ldots e_q^{\alpha_q} \neq \omega_1^{j_1} \ldots \omega_p^{j_p} e_1^{\beta_1} \ldots e_q^{\beta_q} ,$$

in the semigroup F, i.e.

$$\omega_1^{i_1} \ldots \omega_p^{i_p}(e_1^{\alpha_1}, \ldots, e_q^{\alpha_q}) \neq \omega_1^{j_1} \ldots \omega_p^{j_p}(e_1^{\beta_1}, \ldots, e_q^{\beta_q})$$

in the Ω-algebra F.

This proves that (*′), (**) and (***) is an axiom system for the set of identities which are satisfied in all Ω-algebras belonging to $\underline{C}_{r,m}(\Omega)$.

1.2 $r \geq 2$ and $\Omega \neq \Omega(1)$. We shall give an example of an Ω-algebra which does not belong to $\underline{C}_{r,m}(\Omega)$, although it satisfies all the identities (*´), (**) and (***).

Let $\omega \in \Omega(n+1)$, where $n \geq 1$, and let i be the least positive integer such that $in+1-r = p \geq 0$. Thus, $1 \leq i < r$. Let $E = \{e_1,\ldots,e_{rn},e\}$ be a set with rn+1 distinct elements and let A be the Ω-algebra with the presentation

$$\langle E; \omega^i(e_1,\ldots,e_p,e^r) = \omega^r(e_1,\ldots,e_{rn},e)\rangle_{(*'),(**),(***)}$$

where the indices (*´), (**),(***) mean that A satisfies all the identities (*´), (**), (***).

In algebra A the following inequality holds:

$$\omega^i(e_1,\ldots,e_p,e^r) \neq \omega^{r+m}(e_1,\ldots,e_{rn},e^{1+mn}) \,, \tag{6}$$

for neither the left nor right hand side allows a proper transformation by (*´) and, by applying defining relation on $\omega^i(e_1,\ldots,e_p,e^r)$ we get $\omega^r(e_1,\ldots,e_{rn},e)$, so we can only turn to $\omega^i(e_1,\ldots,e_p,e^r)$. But, if we assume that $A \in \underline{C}_{r,m}(\Omega)$, i.e. that A is an Ω subalgebra of semigroup $S \in \underline{C}_{r,m}$, then we would have:

$$\begin{aligned}
&\omega^i(e_1,\ldots,e_p,e^r) = \bar{\omega}^i e_1\ldots e_p e^r = \\
&= \bar{\omega}^i e_1\ldots e_p e^{r+mn} = \omega^i(e_1,\ldots,e_p e^r)e^{mn} = \\
&= \omega^r(e_1,\ldots,e_p,e)e^{mn} = \bar{\omega}^r e_1\ldots e_p e^{1+mn} = \\
&= \bar{\omega}^{r+m} e_1 \ldots e_p e^{1+mn} = \omega^{r+m}(e_1,\ldots,e_p,e^{1+mn}) \,.
\end{aligned}$$

This example shows that, if $r \geq 2$ and $\Omega \neq \Omega(1)$, then $\underline{C}_{r,m}(\Omega)$ is a proper quasi variety.

1.3 $r = 1$. Let A be an Ω-algebra, and let Ω' be a subset of Ω such that different operators of Ω induce different operations on A, and for every $\omega \in \Omega(n)$, there is an $\omega' \in \Omega'(n)$ such that ω and ω' induce the same operation on A. Then, the Ω-algebra A is an Ω-subalgebra of a semigroup S iff the corresponding restricted

Ω' -algebra is an Ω'-subalgebra of S. Morover, (A,Ω) satisfies the identity $(*')$, $(**)$ and $(***)$ iff (A,Ω') satisfies the same identities. Therefore, we can assume that Ω is a set of finitary operations on A, i.e. $\Omega \subseteq O(A)$.

Let L be the subsemigroup of $O(A)$ generated by Ω and let an Ω-algebra satisfy $(*')$, $(**)$, $(***)$. Then, the L-algebra A satisfy the same propositions and by the Theorem 3. the L-algebra is an L-subalgebra of a semigroup $M \in \underline{C}_{1,m}$, hence, we obtain that the given Ω-algebra A is an Ω-subalgebra of M.

1.4 $r \geq 2$ and $\Omega = \Omega(1)$. In this case an Ω-algebra satisfies all the idenitities $(*')$, $(**)$ and $(***)$ iff the semigroup L of transformations (generated by Ω) belongs to $\underline{C}_{r,m}$. By the Theorem 2. we have that if an Ω-algebra satisfies all the identities $(*')$, $(**)$ and $(***)$, then it is an Ω-subalgebra of a semigroup $S \in \underline{C}_{r,m}$. Therefore, $\underline{C}_{r,m}(\Omega)$ is, in this case, a variety.

Thus, the proof of Theorem 1. is completed, i.e. it is reduced to Theorems 2. and 3..

2. P r o o f of Theorem 2. If r=1, then Theorem 2. is a corollary of Theorem 3.. Thus, we have to consider only the case $r \geq 2$.

We may assume that L is a submonoid of $T_A = O_1(A)$, for if it is not we can add to L the identity transformation $\varepsilon_A : a \mapsto a$.

Let B be the monoid in the veriety $\underline{C}_{r,m}$, which is freely generated by A, i.e. the elements of S are "commutative product of powers" $a_1^{\alpha_1} a_2^{\alpha_2} \dots a_q^{\alpha_q}$, where $a_1, \dots, a_q \in A$, $a_i \neq a_j$ for $i \neq j$ and $\alpha_\nu \geq 0$.

$$a_1^{\alpha_1} a_2^{\alpha_2} \dots a_q^{\alpha_q} = a_1^{\beta_1} a_2^{\beta_2} \dots a_q^{\beta_q}$$

iff

$$(\forall \nu \in \{1,2,\dots,q\})(\alpha_\nu = \beta_\nu \text{ or } (\alpha_\nu, \beta_\nu \geq r \text{ and } \alpha_\nu \equiv \beta_\nu (\mathrm{mod}\, m)).$$

Let C be the direct product of L and B. If $u = (\phi, a_1^{\alpha_1} \dots a_q^{\alpha_q})$ then we denote u by $\phi\underline{a}$, where $\underline{a} = a_1^{\alpha_1} \dots a_q^{\alpha_q}$.

If $u = u'a$, $v = \phi u'a'$, $u' \in C$, $a' = \phi(a)$, then we say that (u,v) and (v,u) are two pairs of neighbours. Two elements u,v from C are called equivalent, which is denote by $u \approx v$, iff there is a sequence $u_o, u_1, \ldots, u_k$ of elements of C such that $u = u_o$, $v = u_k$, $k \geq 0$ and (u_{i-1}, u_i) is a pair of neighbours for each $i \in \{1,\ldots,k\}$. Obviously, $\approx$ is a congruence on C. Denote by M the corresponding factor monoid $C_{/\approx}$.

We can assume that L is a submonoid of M, for we have:

(i′) $\quad \phi,\psi \in L \Rightarrow (\phi \approx \psi \Rightarrow \phi = \psi)$.

If $a = \phi(a')$ then $a \approx \phi a'$, and thus the proof will be completed if we show that the following proposition is satisfied

(ii′) $\quad a, a' \in A \Rightarrow (a \approx a' \Rightarrow a = a')$.

Let $a \in A$ and $u_o, u_1, \ldots, u_k$ be a sequence of elements of C such that $a = u_o$ and (u_{i-1}, u_i) is a pair of neighbours for each $i \in \{1,2,\ldots,k\}$. We are going to show that each u_i has a form $u_i = \phi_i a_i$ where $\phi_i(a_i) = a$. First, this is true for $i=0$, $a = u_o = \varepsilon a$, $\varepsilon(a) = a$. Assume that $u_{k-1} = \phi_{k-1} a_{k-1}$ and $\phi_{k-1}(a_{k-1}) = a$. Then, we have

(I) $\quad u_k = \phi\phi_{k-1} a_k$, $\phi(a_k) = a_{k-1}$, and thus $\phi_{k-1}\phi(a_k) = a$

or

(II) $\quad u_k = \phi_{k-1} a_k$, $\phi_{k-1} = \phi\phi_{k-1}$ $\phi(a_{k-1}) = a_k$ and then

$$\phi_{k-1}(a_k) = \phi_{k-1}\phi(a_{k-1}) = \phi_{k-1}(a_{k-1}) = a.$$

This completes the proof of Theorem 2..

3. P r o o f of Theorem 3. L satisfies the assumptions of Theorem 3. iff $L \cup \{\varepsilon\}$ satisfies them, and thus we can assume that L is a submonoid of $O(A)$.

3.1. Let $\equiv$ be the least congruence on L such that $\bar{L} = L_{/\equiv} \in \underline{C}_{1,m}$. More explicitly, $\equiv$ is defined in the following way:

Let $\phi, \psi \in L$, then, $\phi \equiv \psi$ iff there exist $\phi_1, \ldots, \phi_p \in L$ and nonnegative integers $i_{\nu\lambda}$, $j_{\nu\lambda}$ such that:

(3.1) $\quad i_{\nu\lambda} = j_{\nu\lambda} = 0$ or $(i_{\nu\lambda}, j_{\nu\lambda} \geq 0$ and $i_{\nu\lambda} \equiv j_{\nu\lambda} (\mathrm{mod}\, m))$

and the following equalities are satisfied:

$$\phi=\phi_1^{i_{11}}\phi_2^{i_{12}}\ \dots\ \phi_p^{i_{1p}}$$

$$(3.2)\quad \begin{aligned}\phi_1^{j_{11}}\phi_2^{j_{12}}\dots\phi_p^{j_{1p}} &= \phi_1^{i_{21}}\phi_2^{i_{22}}\dots\ \phi_p^{i_{2p}}\\ &\dots\dots\dots\\ \phi_1^{j_{q-11}}\phi_2^{j_{q-12}}\ \dots\ \phi_p^{j_{q-1p}} &= \phi_1^{i_{q1}}\phi_2^{i_{q2}}\dots\ \phi_p^{i_{qp}}\\ \phi^{j_{q1}}\phi^{j_{q2}}\ \dots\ \phi^{j_{qp}} &= \psi.\end{aligned}$$

From the given definition immediately follows

3.1.1. $\phi \in L(n')$, $\psi \in L(n'')$, $\phi \equiv \psi \Rightarrow n' \equiv n''(\mathrm{mod}\, m)$.

3.1.2. $\phi \in L(1)$, $\psi \in L$, $\phi \equiv \psi \Rightarrow \phi = \psi$ (Thus, we assume that $L(1) \subseteq \bar{L} = L_{/\equiv}$).

Now, we are going to show that if $\phi \equiv \psi$, then ϕ and ψ have the same action to "similar sequences".

3.1.3. Let $\phi \equiv \psi$, $\phi \in L(n'+1)$, $\psi \in L(n''+1)$ and $\alpha_1,\dots,\alpha_q,\beta_1,\dots$ $\dots,\beta_q$ are such that $\alpha_\nu,\beta_\nu > 0$, $\alpha_\nu \equiv \beta_\nu$ (mod m)

$$(3.3)\qquad \alpha_1 +\dots+ \alpha_q = n'+1,\ \beta_1+\dots+ \beta_q = n'' + 1.$$

Then,

$$(3.4)\qquad \phi(x_1^{\alpha_1},\dots,x_q^{\alpha_q}) = \psi(x_1^{\beta_1},\dots,x_q^{\beta_q})$$

is an identity equation on A.

P r o o f. If $n'=0$ or $n''=0$, then by 3.1.2. $\phi=\psi$. Thus, we can assume that $n' > 0$ and $n'' > 0$. Let (3.2) be satisfied and let $\phi_\nu \in L(n_\nu+1)$. From $n' > 0$ and $n'' > 0$ it follows that for each μ there exists a λ such that $j_{\mu\lambda} > 0$ and $n_\lambda > 0$. We can assume that $j_{11} > 0$, $n_1 > 0$. Let s_1 be the least nonnegative integer such that

$$(3.5)\qquad 1+j_{11}n_1+s_1mn_1+j_{12}n_2+\dots+j_{1p}n_p-(\beta_1+\beta_2+\dots+\beta_q) = t_1 \geq 0 .$$

Then

$$t_1 \equiv 1+j_{11}n_1+j_{12}n_2+\dots+j_{1p}n_p-(\beta_1+\dots+\beta_q)\ (\mathrm{mod}\, m)$$

$$\equiv 1+i_{11}n_1+i_{12}+n_2+\ldots+i_{1p}n_p-(\alpha_1+\ldots+\alpha_q)\,(\mathrm{mod}\,m) \equiv 0\,(\mathrm{mod}\,m).$$

Now, by (*) we have:

$$\begin{aligned}\phi(x_1^{\alpha_1},\ldots,x_q^{\alpha_q}) &= \phi_1^{i_{11}}\ldots\phi_p^{i_{1p}}(x_1^{\alpha_1},\ldots,x_q^{\alpha_q})\\ &= \phi_1^{j_{11}+s_1m}\,\phi_2^{j_{12}}\ldots\phi_p^{j_{1p}}(x_1^{\beta_1+t_1},x_2^{\beta_2},\ldots,x_q^{\beta_q}).\end{aligned}$$

If $j_{2\lambda_2}$, $n_{\lambda_2}>0$, then in the same way we obtain:

$$\phi(x_1^{\alpha_1},\ldots,x_q^{\alpha_q}) = \phi_1^{j_{21}}\ldots\phi_p^{j_{2p}}\phi_{\lambda_2}^{s_2m}(x_1^{\beta_1+t_2},x_2^{\beta_2},\ldots,x_q^{\beta_q})$$

where s_2 is chosen in a similar way as s_1. Finally, we should obtain

$$\phi(x_1^{\alpha_1},\ldots,x_q^{\alpha_q}) = \psi(x_1^{\beta_1},\ldots,x_q^{\beta_q}).$$

3.1.4. If $\phi\equiv\psi$ and $\phi,\psi\in L(n)$ then $\phi=\psi$.

P r o o f. This is an immediate corollary from 3.1.3..

Further on, if $\phi\in L$ then by $\bar{\phi}$ shall be denoted the element of $\bar{L}=L_{/\equiv}$ such that $\phi\in\bar{\phi}$.

3.2. As in 2, denote by B the monoid in the variety $\underline{C}_{1,m}$ which is freely generated by A, and by C the direct product $\bar{L}\times B$. An element $u=(\bar{\phi},a_1^{\alpha_1}\ldots a_q^{\alpha_q})$ shall be $\bar{\phi}\underline{a}$. The relation of neighbourhoodness shall also be defined in the same way. Namely, if

$$u=u'a,\quad v=\bar{\phi}u'a_1a_2\ldots a_n,\ \phi\in L(n) \text{ and } a=\phi(a_1,\ldots,a_n),$$

then (u,v) and (v,u) are the pairs of neighbours generated by ϕ. The relation $\approx$ is the reflexive and transitive extension of the relation of neighbourhoodness; $\approx$ is a congruence on C. Denote the factor monoid by M.

If $\phi,\psi\in L$ then $\bar{\phi}\approx\bar{\psi}$ iff $\phi\equiv\psi$, and thus $\bar{L}\subseteq M$. By 3.1.2. we have $L(1)\subseteq M$. In further considerations, we are going to prove the following statement:

(ii´) $a,a'\in A \Rightarrow (a\approx a' \Leftrightarrow a=a')$,

which implies (ii), and as we have $a=\phi(a_1,\dots,a_n) \Rightarrow a \approx \bar{\phi} a_1 \dots a_n$ this will complete the proof of Theorem 2.

3.3. In order to prove statement (ii´), we shall consider a special subset T of C, and a mapping $u \mapsto [u]$ from T into A. If $u \in T$ then u is called a "term", and $[u]$ the "value" of u.

Let $u=\bar{\phi} a_0 a_1 \dots a_p \in C$, $\phi \in L(n+1)$, $a_\nu \in A$ be such that $n \equiv p(\mathrm{mod}\, m)$. Then, $u \in T$ iff a) $n \geq 1$, or b) $n=0$, and there is a decomposition $\phi=\phi_0\phi_1\dots\phi_p$ such that

$$\phi_0(a_0) = \phi_1(a_1) = \dots = \phi_p(a_p) = a.$$

In case a), there exist nonnegative integers i,j such that

$$(im+1)n+1 = jm+p+1$$

and then we put

$$[u] = \phi^{im+1}(a_0^{jm+1}, a_1, \dots, a_p) .$$

In case b), value $[u]$ is defined by $[u]=a$.

The value $[u]$ of a term u of form a), does not depend on i,j or on ϕ by (*) and 3.1.3.. But, we have to show that the same is true for a term of a form b).

Namely, if it is possible for ϕ to have another decomposition $\phi=\psi_0\psi_1\dots\psi_q$ such that

$$\psi_0(b_0) = \psi_1(b_1) = \dots = \psi_q(b_q) = b ,$$

where $a_0 a_1 \dots a_p = b_0 b_1 \dots b_q$ in B, we have to show that $a=b$. First, we can assume that $p=q$ and that $a_\nu=b_\nu$. Then, we have

$$\begin{aligned} a &= \phi_0(a_0) = \phi_0^m\phi_0(a_0) = \phi_0^m\phi_1(a_1) = \dots = \phi_0^m\phi_1^m\dots\phi_p^m\phi_0(a_0) = \\ &= \psi_0^m\psi_1^m\dots\psi_p^m\phi_0^p\phi_0(a_0) = \phi_1\psi_0^m\psi_1^m\dots\psi_p^m\phi_0^p(a_1) = \\ &= \phi_1\psi_0\psi_1^{m-1}\dots\psi_p^m\phi_0^p\psi_0(a_0) = \phi_1\psi_0\psi_1^{m-1}\dots\psi_p^m\phi_0^{p-1}\phi_0(a_0) = \\ &= \dots = \phi_1\phi_2\dots\phi_p\psi_0^p\psi_1^{m-1}\dots\psi_p^{m-1}\phi_0(a_0) = \psi_0^{p+1}\psi_1^m\dots\psi_p^m(a_0) = \\ &= \psi_0\psi_1^m\dots\psi_p^m(a_0) = \psi_1^{m+1}\psi_2^m\dots\psi_p^m(a_1) = \psi_2^{m+1}\psi_3^m\dots\psi_p^m(a_2) = \\ &= \psi_p^{m+1}(a_p) = \psi_p(a_p) = b. \end{aligned}$$

Thus, the value $[u]$ of a term u is uniquely determined.

Now, we shall state some propositions concerning terms and values of terms.

3.3.1. If $\bar{\phi}\underline{a} \in T$ and $\phi \in L(sm+1)$ for some $s \geq o$ then $\overline{\phi\psi}\underline{a} \in T$ and

$$[\bar{\phi}[\bar{\psi}\underline{a}]] = [\overline{\phi\psi}\underline{a}] .$$

3.3.2. If $\bar{\phi}\underline{a}a \in T$ and $a=\psi(a_1,\ldots,a_n)$ then $\overline{\phi\psi}\underline{a}a_1 \ldots a_n \in T$ and

$$[\bar{\phi}\bar{\psi}\underline{a}\ a_1\ldots a_n] = [\bar{\phi}\underline{a}a].$$

3.3.3. If $\overline{\phi\psi}\underline{a}b_1\ldots b_n \in T$ and $\psi(b_1,\ldots,b_n) = a$ then $\overline{\phi\psi^m}\underline{a}a \in T$ and

$$[\overline{\phi\psi}\underline{a}b_1\ldots b_n] = [\overline{\phi\psi^m}\underline{a}a] .$$

The proofs of 3.3.1. and 3.3.2. are straightforward and will not be given explicitly. If $\phi\psi$ is not unary, then 3.3.3. is a corollary of 3.3.2., and we are going to consider only the case when $\phi,\psi \in L(1)$:

Assume that $\overline{\phi\psi^i}\underline{a}b_1 \in T$ and $[\overline{\phi\psi^i}\underline{a}b_1] = d$, $i \geq 1$. Then,

$\phi\psi^i=\phi_o\phi_1\ldots\phi_p$, $\underline{a}b_1 = a_1 \ldots a_p b_1$, $p \equiv 0 (\mathrm{mod}\, m)$ $b_1, a_\nu \in A$, $\psi(b_1) = a$,

$$d = \phi_o(b_1) = \phi_1(a_1) = \ldots = \phi_p(a_p) ,$$

and

$$\psi(d) = \phi_o(\psi(b_1)) = \psi\phi_1(a_1) = \ldots = \psi\phi_p(a_p) ,$$

where we obtain

$$[\phi_o\phi_1\ldots\phi_p\psi^p\underline{a}a] = [\phi\psi^i\underline{a}a] = \psi(d)$$

for $a = \psi(b_1)$, and $\psi^{i+p} = \psi^i$. From $\overline{\phi\psi^i}\underline{a}a \in T$ and $a = \psi(b_1)$, by 3.3.2., it follows that $\phi\psi^{i+1}\underline{a}b_1 \in T$ and $[\phi\psi^i\underline{a}a] = [\overline{\phi\psi^{i+1}}\underline{a}b_1]$.

Thus we have

$$[\overline{\phi\psi^m}\underline{a}a] = [\overline{\phi\psi^{m+1}}\underline{a}b_1] = [\overline{\phi\psi}\underline{a}b_1].$$

4.3. Here statement (ii´) (from the end of 3.2.) will be shown, and this will complete the proof of Theorem 3.

First, we prove that

4.3.1. If $a=u_o, u_1, \ldots, u_p$ is a sequence of elements of C such that $p \geq 0$ and u_{i-1}, u_i is a pair of neighbours generated by ϕ_i for each $i \in \{1,\ldots,p\}$, then $\phi_1^m \ldots \phi_q^m u_i \in T$ for each $i \in \{1,\ldots,p\}$ and:

$$(3.6) \qquad [\phi_i^m a] = [\phi_1^m \ldots \phi_i^m u_i] = a.$$

P r o o f. Assume that (3.6) is true, and that $i < p$. Then:

(I) $u_i = ub, \; u_{i+1} = \phi u b_1 \ldots b_n$,

or

(II) $u_i = \phi u b_1 \ldots b_n, \quad u_{i+1} = ub$,

where $\phi = \phi_{i+1}$, $b = \phi(b_1,,,.b_n)$, $u \in C$.

In case (I), by 3.3.2. we have that

$$[\phi_1^m \ldots \phi_i^m \phi^m u_{i+1}] = [\phi_1^m \ldots \phi_i^m \phi u b_1 \ldots b_n] = [\phi_1^m \ldots \phi_i^m u_i] = a,$$

and by 3.3.1.

$$[\phi^m a] = [\phi^m [\phi_1^m \ldots \phi_i^m u_i]] = [\phi^m \phi_1^m \ldots \phi_i^m ub] =$$
$$= [\phi^m \phi_1^m \ldots \phi_i^m \phi u b_1 \ldots b_n] = [\phi^m \phi_1^m \ldots \phi_i^m u_{i+1}] = a.$$

In case (II), we have:

$a = [\phi_1^m \ldots \phi_i^m u_i] = [\phi_1^m \ldots \phi_i^m \phi u b_1 \ldots b_n]$ and by 3.1.3. this implies that

$$a = [\phi_1^m \ldots \phi_i^m \phi^m ub] = [\phi_1^m \ldots \phi_i^m \phi^m u_{i+1}].$$

We also have

$$[\phi^m a] = \underline{\phi}^m [\phi_1^m \ldots \phi_i^m u_i]] = [\phi_1^m \ldots \phi_i^m \phi^m \phi u b_1 \ldots b_n] =$$
$$= [\phi_1^m \ldots \phi_i^m \phi u b_1 \ldots b_n] = [\phi_1^m \ldots \phi_i^m u_i] = a$$

and this complete the proof of 4.3.1.

P r o o f. of 3.2. (ii´)

Assume that $a, a' \in A$ and $a \approx a'$. Then, there exists a sequence of elements $u_o, u_1, \ldots, u_p$ of C such that $a = u_o$, $a' = u_p$ and (u_{i-1}, u_i) is a pair of neighbours generated by $\phi_i \in L$. By 4.3.1. we have

$$a = [\phi_1^m \ldots \phi_p^m a'] \ , \quad a = [\phi_1^m a] = \ldots = [\phi_p^m a],$$

and also

$$a' = [\phi_p^m \ldots \phi_1^m a], \quad a' = [\phi_1 a'] = \ldots = [\phi_p a'],$$

which implies that $a = a'$.

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REZIME

PODALGEBRE KOMUTATIVNIH POLUGRUPA KOJE ZADOVOLJAVAJU ZAKON $x^r = x^{r+m}$.

Algebra tipa Ω sa nosačem A naziva se Ω-podalgebra polugrupe S ako je $A \subset S$ i ako postoji preslikavanje $\omega \mapsto \bar{\omega}$ Ω u S takvo da je

$$\omega(a_1, \ldots, a_n) = \bar{\omega} a_1 \ldots a_n$$

za svaku n-arnu operaciju $\omega \in \Omega$ i niz elemenata $a_1, \ldots, a_n$ iz A. Ako je K klasa polugrupa tada sa $K(\Omega)$ označavamo klasu Ω-algebri koje su podalgebre polugrupa koje pripadaju K. Ako je K varijetet polugrupa, tada je $K(\Omega)$ kvazivarijetet Ω-algebri.

U ovom radu daju se potrebni i dovoljni uslovi da $\underline{C}_{r,m}(\Omega)$ bude varijetet (Teorema 1.). U teoremama 2. i 3. dat je opis polugrupa operacija koje se mogu potopiti u polugrupe iz $\underline{C}_{r,m}$.

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TRANSITIVE n-ARY RELATIONS AND CHARACTERIZATIONS OF GENERALIZED EQUIVALENCES*)

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Pickett |2| defines generalized equivalence relations and relates them to the partitions of type n, given by Hartmanis |1|. In this article several types of generalized reflexive, symmetric and also transitive relations are defined and properties and connections between some of these relations are given. Finally, some characterization theorems for generalized equivalence relations are proved.

*

1. (n+1)-ary relation R on the set $S \neq \emptyset$ is (i,j)-reflexive, $i \neq j$, $i,j \in \{1,\ldots,n+1\}$, iff

$$(\forall a_1,\ldots,a_{j-1},a_{j+1},\ldots,a_{n+1} \in S)\ ((a_1^{i-1}, a_i, a_{i+1}^{j-1}, a_i, a_{j+1}^{n+1}) \in R).^{1)}$$

R is reflexive iff it is (i,j)-reflexive for all $i,j \in \{1,\ldots,n+1\}$, $i \neq j$.[2)]

2. (n+1)-ary relation R on S is π-symmetric, $\pi \in \{1,\ldots,n+1\}!$,[3)] iff

$$(\forall a_1,\ldots,a_{n+1} \in S)((a_1^{n+1}) \in R \Rightarrow (a_{\pi(1)},\ldots,a_{\pi(n+1)}) \in R).$$

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*) Presented april, 27, 1981. 1) a_p^q stands for $a_p, a_{p+1}, \ldots, a_{q-1}, a_q$, and denotes an empty syllable when $q < p$; consequently a_p^p is a_p, and instead of a, a,...,a (n times), we write $\overset{n}{a}$; $\overset{0}{a}$ is, clearly, empty.

2) In |2| (n+1)-ary (1,n+1)-reflexive relation is called "reflexive"; in |3| the term "strongly reflexive" is used for the reflexive relations.

3) If M finite, M! is a set of all permutations on M.

(|2|) R is <u>symmetric</u> iff it is π-symmetric for all $\pi \in \{1,\ldots,n+1\}!$.

3. Let R be (n+1)-ary relation on S and ${}^1a_1,\ldots,{}^1a_{n+1},\ldots,{}^ka_1,\ldots,{}^ka_{n+1}$, $b_1,\ldots,b_{n+1}$ variables. Let also

1) $k \in N \setminus \{1\}$;
2) α is the k-ary relation on the set R; and
3) (b_1^{n+1}) is taken by the given nullary operation in $\{{}^1a_1,\ldots,{}^1a_{n+1},\ldots,{}^ka_1,\ldots,{}^ka_{n+1}\}^{n+1}$.

R now belongs to the <u>class of transitive relations</u> iff the following implication is satisfied:

$$({}^1a_1^{n+1}) \in R \wedge \ldots \wedge ({}^ka_1^{n+1}) \in R \wedge \wedge (({}^1a_1^{n+1}),\ldots,({}^ka_1^{n+1})) \in \alpha \Rightarrow (b_1^{n+1}) \in R, \tag{1}$$

for all ${}^1a_1,\ldots,{}^1a_{n+1},\ldots,{}^ka_1,\ldots,{}^ka_{n+1} \in S$.[1)]

In this article we shall be concerned with some relations belonging to this class, with k=2, and k=n+1.

3_1) (n+1)-ary relation R on S is <u>iA_1-transitive</u>, $i \in \{1,\ldots,n\}$. iff

$$(\forall a_o,\ldots,a_{n+1} \in S)((a_o^{i-1},a_i,a_{i+1}^n) \in R \wedge (a_1^{i-1},a_i,a_{i+1}^{n+1}) \in R \Rightarrow \Rightarrow (a_o^{i-1},\, a_{i+1}^{n+1}) \in R). \tag{2}$$

(n+1)-ary relation R on S is iA_1^*-transitive, $i \in \{1,\ldots,n\}$, iff

$$(\forall a_o,\ldots,a_{n+1} \in S)((a_o^{i-1},a_i a_{i+1}^n) \in R \wedge (a_1^{i-1},a_i a_{i+1}^{n+1}) \in R \wedge \wedge (a_j \neq a_i, \text{ for } j \in \{1,\ldots,n\} \setminus \{i\}) \Rightarrow (a_o^{i-1},a_{i+1}^{n+1}) \in R). \tag{3}$$

(n+1)-ary relation R on S is $i\bar{A}_1$-transitive, *) $i \in \{1,\ldots,n\}$, iff

$(\forall a_o,\ldots,a_{n+1} \in S)((a_o^{i-1},a_i a_{i+1}^n) \in R \wedge (a_1^{i-1},a_i a_{i+1}^{n+1}) \in R \wedge$

1) In the transitivities considered here, α shall always be such that for n=1 (1) reduces to the usual transitivity low.

*) In |2|: "transitive" stands for "$n\bar{A}_1$-transitive.

(4) $$(a_j \neq a_k,\ \text{for } j=k,\ j,k \in \{1,\dots,n\}) \Rightarrow (a_o^{i-1}, a_{i+1}^{n+1}) \in R)\ .$$

3_2) (n+1)-ary relation R on S is $i\bar{A}_2$-transitive,[2] $i \in \{1,\dots,n\}$ iff

$$(\forall a_o,\dots,a_{n+1} \in S)((a_o, a_1^{i-1}, a_i, a_{i+1}^{n}) \in R \wedge (a_i, a_1^{i-1}, a_{i+1}^{n+1}) \in R \wedge$$

$$(a_j \neq a_k,\ \text{for } j \neq k, j,k \in \{1,\dots,n\}) \Rightarrow (a_o, a_1^{i-1}, a_{i+1}^{n+1}) \in R).$$

3_3) (n+1)-ary relation R on S is $i\bar{M}_1$-transitive, $i \in \{2,\dots,n+1\}$, iff

$$(\forall a_1,\dots,a_{i-1},a_{i+1},\dots,a_{n+1},x_i^{(1)},\dots,x_i^{(n)},y \in S)((x_i^{(j)} \neq$$

$$\neq x_i^{(k)},\ \text{for } j \neq k,\ j,k \in \{1,\dots,n\}) \wedge \bigwedge_{s=1}^{n} (a_1^{i-1}, x_i^{(s)}, a_{i+1}^{n+1}) \in R \wedge$$

$$(x_i^{(1)},\dots,x_i^{(n)},y) \in R \Rightarrow (a_1^{i-1}, a_{i+1}^{n+1}, y) \in R\ .$$[3]

3_4) (n+1)-ary relation R on S is $i\bar{M}_2$-transitive, $i \in \{2,\dots,n+1\}$ iff

$$(\forall a_1,\dots,a_{i-1},a_{i+1},\dots,a_{n+1},x_i^{(1)},\dots,x_i^{(n)},y \in S)((x_i^{(j)} \neq$$

$$\neq x_i^{(k)},\ \text{for } j \neq k,\ j,k \in \{1,\dots,n\}) \wedge \bigwedge_{s=1}^{n} (a_1^{i-1}, x_i^{(s)}, a_{i+1}^{n+1}) \in R \wedge$$

$$(x_i^{(1)},\dots,x_i^{(n)},y) \in R \Rightarrow (a_1^{i-1}, y, a_{i+1}^{n+1}) \in R\ ).$$

REMARK.

In the case n=1 all notions defined in 1), 2) and 3) reduce to the usual binary notions.

4_1) |1| For set S with at least n elements, the family P_n of subsets of S is a partition of type n, iff (1) each member of P_n has at least n elements and (2) each n different elements

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2) The notion of an $i\bar{A}_2$-transitive relation is from |4|. It is obvious that one can define transitive relations without or with one star in 3_2), 3_3) and 3_4) as in 3_1). It has not been done here since the purpose of this atricle is to treat the transitivities connected with generalized equivalence relations.

3) M-transitivities appeared in investigation of generalized orderings.

$\bigwedge_{i=1}^{n} P_i$ denotes a logical conjunction $P_1 \wedge \dots \wedge P_n$.

S belong to exactly one member of P_n.

4_2) |2| (n+1)-ary relation E_n on S is a generalized equivalence relation on S iff it satisfies:

E1n: (1,n+1)-reflexivity,

E2n: symmetry, and

E3n: $n\bar{A}_1$-transitivity.

4_3) In |2| it is shown that (n+1)-ary (i.e. generalized) equivalence relation E_n on S induces on S a partition of type n, and contrary, that each partition of type n on S can be connected with the generalized, (n+1)-ary equivalence relation on the same set.

* *

PROPOSITION 1. *If* (n+1)-*ary relation* R *on* S *is* (1, i+1)-*reflexive and* $i\bar{A}_1$ -*transitive, then it is* $i\bar{A}_2$ -*transitive.*

P r o o f. Let

a) $(x_o, x_1^{i-1}, x_i, x_{i+1}^n) \in R$ and

b) $(x_i, x_1^{i-1}, x_{i+1}^n, x_{n+1}) \in R$,

where $x_i \neq x_j$ for $i \neq j$, $i,j \in \{1,\dots,n\}$. Then

c_1) $(x_{i-1}, x_i, x_1^{i-2}, x_{i-1}, x_{i+1}^n) \in R$ ((1,i+1)-reflexivity).

From (c_1) and (b), by $i\bar{A}_1$-transitivity, it follows

b_1) $(x_{i-1}, x_i, x_1^{i-2}, x_{i+1}^n, x_{n+1}) \in R$. Applying $i\bar{A}_1$-transitivity on (b_1) and

c_2) $(x_{i-2}, x_{i-1}, x_i, x_1^{i-3}, x_{i-2}, x_{i+1}^n) \in R$ ((1,n+1)-reflexivity),

we get

b_2) $(x_{i-2}, x_{i-1}, x_i, x_1^{i-3}, x_{i+1}^{n+1}) \in R$.

This procedure leads to the conclusion

($\bar{b}$) $(x_1^{i-1}, x_i, x_{i+1}^{n+1}) \in R$. Finally, from (a) and ($\bar{b}$), by $i\bar{A}_1$-transitivity it follows that

$(x_o, x_1^{i-1}, x_{i+1}^{n+1}) \in R$, which was to be proved.

COROLLARY 2. *$(n+1)$-ary $(1,n+1)$-reflexive and $n\bar{A}_1$- transitive relation* R *on* S *is* $n\bar{A}_2$ *-transitive.*

REMARK 2.

$(1,n+1)$-reflexivity and $n\bar{A}_2$-transitivity do not imply $n\bar{A}_1$-transitivity, which can be shown by the following example, for n=2. R is a ternary relation on $\{a,b,c,d\}$ consisting of all triples with equal first and third coordinates and of (a,b,c), (c,b,d), (a,b,d) and (b,d,a). It is easy to check that R satisfies $(1,3)$-reflexivity and $2\bar{A}_2$-transitivity, but that it is not $2\bar{A}_1$-transitive.

The following corollary is a consequence of the proof of Proposition 1.

COROLLARY 3. *$(n+1)$-ary $(1,i+1)$ -reflexive and $i\bar{A}_1$ - transitive relation* R *on* S *satisfies the property*

$$(\gamma_i) : (\forall a_1, \ldots, a_{n+1} \in S)((a_j \neq a_k, j \neq k,\ j,k \in \{1,\ldots,n\}) \wedge$$

$$(a_1^{n+1}) \in R \Rightarrow (a_{\gamma_i(1)}, \ldots, a_{\gamma_i(n+1)}) \in R),$$

$$\gamma_i = (i,1,\ldots,i-1,i+1,\ldots,n+1) \in \{1,\ldots,n+1\}!.$$

PROPOSITION 4. *If* $(n+1)$*-ary* $(1,i+1)$*-reflexive and* $i\bar{A}_2$ *transitive relation* R *on* S *satisfies* (γ_i)*, then* R *is* $i\bar{A}_1$ *-transitive* $(i \in \{1,\ldots,n\})$.

P r o o f. If

$$(x_o^{i-1}, x_i, x_{i+1}^n) \in R \text{ and } (x_1^{i-1}, x_i, x_{i+1}^{n+1}) \in R\ ,\ x_i \neq x_j \text{ for } i \neq j,$$

$i,j \in \{1,\ldots,n\}$, then by (γ_i) it follows that

$$(x_o^{i-1}, x_i, x_{i+1}^n) \in R \quad\text{and}\quad (x_i x_1^{i-1}, x_{i+1}^{n+1}) \in R\ .$$

Thereby $i\bar{A}_2$-transitivity implies

$(x_o^{i-1}, x_{i+1}^{n+1}) \in R$, completing the proof of the lemma.

Proposition 1, Corollary 3 and Proposition 4 imply the following proposition.

PROPOSITION 5. *If* $(n+1)$*-ary* $(1,n+1)$ *-reflexive relation* R *on* S *satisfies* (γ_i), *then* R *is* $i\bar{A}_2$ *-transitive iff it is* $i\bar{A}_1$ *-transitive*, $(i \in \{1,\dots,n\})$.

PROPOSITION 6. *If* $(n+1)$*-ary relation* R *on* S *satisfies* $(i+1)\bar{M}_1$*-transitivity* $(i=1,\dots,n)$, $(j,i+1)$*-reflexivity for all* $j \in \{2,\dots,i\}$ *and* $(i+1,k)$ *-reflexivity for all* $k \in \{i+2,\dots,n+1\}$, *then* R *satisfies* $i\bar{A}_1$*-transitivity.*

P r o o f. Let $(x_o^n) \in R$ and $(x_1^{n+1}) \in R$, $x_u \neq x_v$, $u \neq v$, $u,v \in \{1,\dots,n\}$. Then

$(x_o,x_1,\dots,x_{i-1},x_1,x_{i+1},\dots,x_n) \in R$ ((2,i+1)-reflexivity),

$(x_o,x_1,\dots,x_{i-1},x_2,x_{i+1},\dots,x_n) \in R$ ((3,i+1)-reflexivity),

. . . .

$(x_o,x_1,\dots,x_{i-1},x_{i-1},x_{i+1},\dots,x_n) \in R$ ((i,i+1)-reflexivity),

$(x_o,x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_n) \in R$ (by assumption),

$(x_o,x_1,\dots,x_{i-1},x_{i+1},x_{i+1},\dots,x_n) \in R$ ((i+1,i+2)-reflexivity),

. . . .

$(x_o,x_1,\dots,x_{i-1},x_n,x_{i+1},\dots,x_n) \in R$ ((i+1,n+1)-reflexivity),

$(x_1,\dots,x_{i-1},x_n,x_{i+1},\dots,x_n,x_{n+1}) \in R$ (by assumption).

Thereby $(i+1)\bar{M}_1$-transitivity implies

$(x_o,x_1,\dots,x_{i-1},x_{i+1},\dots,x_n,x_{n+1}) \in R$, which was to be proved.

COROLLARY 7. *If* $(n+1)$*-ary* $(n+1)\bar{M}_2$*-transitive relation* R *on* S *satisfies* $(i,n+1)$ *-reflexivity for all* $i \in \{2,\dots,n\}$, *then* R *satisfies* $n\bar{A}_1$*-transitivity.*

REMARK 3.

The following example is a reflexive, $n\bar{A}_1$-transitive ternary relation on $\{a,b,c,d,e\}$, which is not $(n+1)\bar{M}_2$-transitive. Let R consist of all triples with at least two equal coordinates and of (a,b,c),(b,a,c),(a,b,d),(b,a,d),(c,d,e) and

(d,c,e). It is obvious that R is reflexive, $2\bar{M}_2$-transitive, but not $3\bar{M}_2$-transitive, since $(a,b,c) \in R$, $(a,b,d) \in R$, $(c,d,e) \in R$, but $(a,b,e) \notin R$.

COROLLARY 8. *If* $(n+1)$-*ary relation* R *on* S *satisfies* $(i+1)\bar{M}_2$-*transitivity* $(i=1,\dots,n)$, $(j,i+1)$-*reflexivity for all* $j \in \{2,\dots,n\}$, $(i+1,k)$ -*reflexivity for all* $k \in \{i+2,\dots,n+1\}$ *and* γ-*symmetry for* $\gamma = (1,\dots,i,n+1,i+1,\dots,n) \in \{1,\dots,n+1\}$! , *then* R *satisfies* $i\bar{A}_1$-*transitivity.*

PROPOSITION 9. *If* $(n+1)$-*ary relation* R *on* S *satisfies* $(1,j)$-*reflexivity for all* $j \in \{2,\dots,i+1\}$, $i\bar{A}_1$- *and* $(i-1)\bar{A}_1$-*transitivity,* $i \in \{1,\dots,n\}$, *then* R *is* $(i-2)\bar{A}_1$-*transitive* $(i-2,i-1,$ $i \in \{1,\dots,n\})$.

P r o o f.

(a) $\quad (a_0^{i-3},a_{i-2},a_{i-1}^{n}) \in R$ and $(a_1^{i-3},a_{i-2},a_{i+1}^{n+1}) \in R$, $a_i \neq a_j$, $i \neq j$, $i,j \in \{1,\dots,n\}$.

i) Suppose first $a_0 \neq a_i$, $i=1,\dots,n$. Then, by using the well known properties of permutations, by Corollary 3 and since R is $i\bar{A}_1$-, $(i-1)\bar{A}_1$-transitive and $(1,i)$-reflexive, (a) implies

$(\bar{a})$ $\quad (a_0^{i-1},a_{i-2},a_i^{n}) \in R$ and $(a_1^{i-1},a_{i-2},a_i^{n+1}) \in R$.

By $(i-1)\bar{A}_1$-transitivity then $(a_0^{i-3},a_{i-1}^{n+1}) \in R$.

(ii) Let $a_0 \neq a_{i-2}$, and

$(\bar{b})$ $\quad (a_{i-2},a_1^{i-3},a_{i-2},a_{i-1}^{n}) \in R$ and $(a_1^{i-3},a_{i-2},a_{i-1}^{n+1}) \in R$.

By using $(1,i)$-reflexivity and by the procedure used in i),

$(\bar{c})$ $\quad (a_{i-2},a_1^{i-3},a_{i-1},a_{i-2},a_i^{n}) \in R$ and

$(\bar{d})$ $\quad (a_1^{i-3},a_{i-1},a_{i-2},a_i^{n+1}) \in R$ . .

From $(\bar{c})$ and $(\bar{d})$, by $(i-1)\bar{A}_1$-transitivity, if follows that

$$(a_{i-2},a_1^{i-3},a_{i-1},a_1^{n+1}) \in R .$$

iii) Let $a_o = a_j$, $j \in \{1,\ldots,i-3,i-1,\ldots,n\}$, and

$$(a_t, a_1^{i-3}, a_{i-2}, a_{i-1}^{t-1}, a_t, a_{t+1}^{n}) \in R \quad \text{and} \quad (a_1^{i-3}, a_{i-2}, a_{i-1}^{t-1}, a_t, a_{t+1}^{n+1}) \in R,$$

where $t < i$ or $t > i$, $t \in \{1,\ldots,n\}$. $(1,t+1)$-reflexivity now gives

$$(a_t, a_1^{i-3}, a_{i-1}, a_i^{t-1}, a_t, a_{t+1}^{n+1}) \in R .$$

i), ii) and iii) prove the proposition.

Using the fact that each permutation on $\{1,\ldots,n+1\}$ can be produced by two cycles $\gamma_{n+1} = (n+1,1,\ldots,n)$ and $\gamma_n = (n+1,\ldots \ldots,n-1,n+1)$, one can easely show that the following propositi- on is a consequence of the previously proved statements.

PROPOSITION 10. *If* $(n+1)$*-ary relation* R *on* S *satisfies* γ_{n+1} *and* γ_n*-symmetry* (γ_{n+1} *and* γ_n *are given above*), *then*

1) R *is reflexive iff it is* $(1,n+1)$*-reflexive ;*

2) R *is* $i\bar{A}_1$ *-transitive,* $i \in \{1,\ldots,n\}$, *iff it is* $n\bar{A}_1$*- transitive ;*

3) R *is* $i\bar{A}_2$ *-transitive,* $i \in \{1,\ldots,n\}$, *iff it is* $n\bar{A}_2$*-transitive ;*

4) R *is* $i\bar{M}_1$ *-transitive,* $i \in \{2,\ldots,n+1\}$, *iff it is* $(n+1)\bar{M}_1$ *-transitive ;*

5) R *is* $i\bar{M}_2$*-transitive,* $i \in \{2,\ldots,n+1\}$, *iff it is* $(n+1)\bar{M}_2$*-transitive ;*

6) R *is* $n\bar{A}_2$*-transitive, iff it is* $n\bar{A}_1$*-transitive.*

PROPOSITION 11. $(n+1)$*-ary relation* R *on* S *is the generalized equivalence relation on* S *in the sense of* 4_2*) iff*

I R *is reflexive ;*

II R *satisfies the property*

$$(\tau) \quad (\forall a_1,\ldots,a_{n+1} \in S)((a_1^{n+1}) \in R \wedge (a_i \neq a_j,\ i \neq j,\ i,j \in \{1,\ldots,n\})$$
$$\Rightarrow (a_{\tau(1)},\ldots,a_{\tau(n+1)}) \in R, \textit{ where } \tau = (n+1,2,\ldots,n,1) ;$$

III R *is* $n\bar{A}_1$*-transitive.*

P r o o f. The only nontrivial part of the proof is the one in which the symmetry has to be proved, under the assumption that R satisfies I, II and III.

Suppose $(a_1^{n+1}) \in R$.

If $a_i = a_j$, for some $i \neq j$, $i,j \in \{1,\ldots,n+1\}$, then this part of the symmetry follows directly from I and the definition of reflexivity.

If there are no equal elements among a_i, $i \in \{1,\ldots,n+1\}$, then Corollary 3, I, and III imply that if

$(a_1^{n+1}) \in R$ then all (n+1)-tiples produced by the cycle $(n,1,\ldots,n-1,n+1)$ also belong to R. Since this cycle and τ produce each permutation on $\{1,\ldots,n+1\}$, using II, we get that for each permutation $\pi \in \{1,\ldots,n+1\}$!

$(a_{\pi(1)},\ldots,a_{\pi(n+1)}) \in R$, proving that T satisfies the symmetry.

There is another characterization of the generalized equivalence relation, depending on reflexivity and two transitivities.

PROPOSITION 12. (n+1)-*ary relation* R *on* S *is a generalized equivalence relation in the sense of* 4_2) *iff*

(α) R *is reflexive* ;

(β) R *is* $(n-1)\bar{A}_1$ *-transitive* ;

(γ) R *is* $n\bar{A}_1$ *-transitive* .

P r o o f. Let $(a_0, a_1^{n-2}, a_{n-1}, a_n) \in R$. By (n-1,n+1)-reflexivity $(a_1^{n-2}, a_{n-1}, a_n, a_{n-1})$ also belongs to R. Thereby, for different $a_1,\ldots,a_{n+1}$, (β) implies that if $(a_1^{n+1}) \in R$, then $(a_1^{n-1}, a_{n+1}, a_n) \in R$. The proof of the symmetry now follows from the procedure used in proving Proposition 11.

The following proposition shows that, assuming (1,n+1)-reflexivity and some of the symmetry, $i\bar{A}_1$-transitivity implies $(i+1)\bar{M}_1$-transitivity. Putting together this statement and Proposition 6., one gets the conditions for the equivalence of these two transitivies.

PROPOSITION 13. *If* $(n+1)$*-ary relation* R *on* S *satisfies* γ_{n+1} *- and* γ_n *-symmetry* ($\gamma_{n+1}=(n+1,1,\dots,n)$, $\gamma_n=(n,1,\dots,n-1,n+1)$), $i\bar{A}_1$ *-transitivity* ($i\in\{1,\dots,n\}$) *and* $(1,n+1)$ *-reflexivity, then* R *satisfies* $(i+1)\bar{M}_1$ *-transitivity.*

P r o o f. 1) R is symmetric, because it satisfies γ_{n+1}- and γ_n- symmetry (see the text preceding Proposition 10.).

2) Let $(a_1^n,x_1)\in R,\dots,(a_1^n,x_n)\in R$, $x_i\neq x_j$ for $i\neq j$, $i,j\in\{1,\dots,n\}$. Now, if there are equal elements among $a_1,\dots,a_n,y$, reflexivity implies $(a_1^n,y)\in R$, proving $(i+1)\bar{M}_1$-transitivity for R.

Assume now that $a_i\neq a_j$. $a_i\neq y$ $i,j\in\{1,\dots,n\}$. Also let $a_i=x_i$, $i=1,\dots,k$, $k\in\{0,\dots,n\}$ (for k=0 there are no such equal elements). Assumption 2) is now given by

(a) $(x_1^k,a_{k+1}^n,x_1)\in R,\dots,(x_1^n,a_{k+1}^n,x_k)\in R$,

(b) $(x_1^k,a_{k+1}^n,x_{k+1})\in R,\dots,(x_1^k,a_{k+1}^n,x_n)\in R$.

From (b), by symmetry, it follows that

$$(x_{k+1},x_1^k,a_{k+1}^n)\in R \text{ and } (x_1^k,a_{k+1}^n,x_t)\in R,\ t\in\{k+2,\dots,n\}.$$

$n\bar{A}_1$ -transitivity now implies

(c) $(x_{k+1},x_1^k,a_{k+1}^{n-1},x_t)\in R$, $t\in\{k+2,\dots,n\}$.

From this, by symmetry and $n\bar{A}_1$-transitivity, it follows that

(d) $(x_{k+2},x_{k+1},x_1^k,a_{k+1}^{n-2},x_v)\in R$, $v\in\{k+3,\dots,n\}$.

Continuing this procedure, we finally get $(x_1^n,a_{k+1})\in R$, i.e.

(e) $(a_{k+1},x_1^n)\in R$.

Since $x_i=a_i, i=1,\dots,k$, from (a), (b) and (e) it follows that $(a_{k+1},a_1^k,x_{k+1}^n)\in R$ and $(a_1^k,x_{k+1}^n,y)\in R$, and applying $n\bar{A}_1$-transitivity, we get

(f) $(a_{k+1},a_1^k,x_{k+1}^{n-1},y)\in R$.

Applying $n\bar{A}_1$-transivity on

$(a_{k+2},a_{k+1},a_1^k,x_1^{n-1}) \in R$, $(a_{k+1},a_1^k,x_1^{n-1},y) \in R$,

(the first (n+1)-tiple is from (d) i.e. from the procedure exposed there, and the second is (f)), we get

(g) $(a_{k+2},a_{k+1},a_1^k,x_{k+1}^{n-2},y) \in R$.

Continuing this procedure, we finally get

$(a_n,a_{n-1},\ldots,a_{k+1},a_1^k,y) \in R$, i.e. $(a_1^n,y) \in R$, proving the statement, if $i=n$.

In the case when $i \neq n$, suppose that

(h) $(a_1^{n-1},x_1,a_{i+1}^{n+1}) \in R,\ldots,(a_1^{i-1},x_n,a_{i+1}^{n+1}) \in R$ and

(i) $(x_1^n,y) \in R$, $x_i \neq x_j$, for $i \neq j$, $i,j \in \{1,\ldots,n\}$.

Since R is symmetric (1)), we have (from (h) and (i))

(j) $(a_1^{i-1},a_{i+1}^{n+1},x_1) \in R,\ldots,(a_1^{i-1},a_{i+1}^{n+1},x_n) \in R$ and

(k) $(x_1^n,y) \in R$, $x_i \neq x_j$, for $i \neq j$, $i,j \in \{1,\ldots,n\}$.

Applying the symmetry and 4), Proposition 10, the proof in this case is the same as the one given for $i=n$.

COROLLARY 14. *If* (n+1)*-ary reflexive relation on* S *satisfies* $(n-1)\bar{A}_1$*- and* $n\bar{A}_1$*-transitivity, then it is* $(n+1)\bar{M}_1$ *-transitive.*

P r o o f. This is a consequence of the two previous propositions.

COROLLARY 15. (n+1) *-ary* (1,n+1)*-reflexive and symmetric relation* R *on* S *is* $n\bar{A}_1$*-transitive iff it is* $(n+1)\bar{M}_1$*-transitive.*

P r o o f. Immediately by Proposition 6. and Proposition 13.

REMARK 4.

Applying Corollary 8., and the first part of the proof of Proposition 13. concerning the symmetry, one can put "$(i+1)\bar{M}_2$-" instead of "$(i+1)\bar{M}_1$-" into the formulation of Proposition 13., which will remain true.

* * *

In this part we shall show that the condition of n different elements in $i\bar{A}_1$-transitivity ((4) in (3_1), given by Pickett |2| can be weakened ((3) in 3_1) , giving something new in the previous characterizations.

The following lemma follows immediately from the definitions of iA_1^*- and $i\bar{A}_1$-transitivity.

LEMMA 16. *a)* iA_1^* *-transitive relation is* $i\bar{A}_1$*-transitive;*

b) For $n=2$ *the relation* R *is* $i\bar{A}_1$*-transitive iff it is* $i\bar{A}_1^*$*-transitive (i.e. these two definitions to not differ).*

REMARK 5.

If we consider reflexive relations, the condition $a_j \neq a_i$ could not be weakened more, since for example, for n=2, from $(a,b,b) \in R$ and $(b,b,c) \in R$, its absence implies $(a,b,c) \in R$. The reflexive relations would thus always consist of all triples ((n+1)- tiples) on the given set.

PROPOSITION 17. $(n+1)$*-ary relation* R *on* S $\|S\| \geq n)$, *is the generalized equivalence relation on* S *in the sense of* $4_2)$ *iff it satisfies:*

(i) for each sequence of n *different elements* $x, y_1, \ldots$ $\ldots, y_{n-1}$ *of* S*, there is* y *in* S, $y = y_{n-1}$*, such that*

$$(x, y_1^{n-1}, y) \in R ;$$

(ii) R *is* γ_{n+1}*-and* γ_n*-symmetric, where*

$\gamma_{n+1} = (n+1,1,\ldots,n)$ *and* $\gamma_n = (n,1,\ldots,n-1,n+1)$ [1)];

(iii) R *is* nA_1^* - *transitive.*

REMARK 6.

For n=1, (i) reduce to the statement that for each $x \in S$, there is $y \in S$, such that $(x,y) \in R$, and this is equivalent to the usual condition $\mathcal{D}R = S$, in the binary case.

Proof of Proposition 17: We have to prove that R satisfies (1,n+1)-reflexivity (the rest is trivial), if it satisfies (i), (ii) and (iii).

Let $x_o,\ldots,x_{n-1} \in S$, $x_i \neq x_j$, $i,j \in \{0,\ldots,n-1\}$. Then by (i), there is $y \in S$, $y \neq x_{n-1}$, and $(x_o^{n-1},y) \in R$. From here, by symmetry, it follows that $(x_1^{n-1},y,x_o) \in R$, and by (iii)

$$(x_o^{n-1},x_o) \in R, \text{i.e. (symmetry)}, \quad (\overset{2}{x}_o,x_1^{n-1}) \in R.$$

From $(x_o^{n-1},y) \in R$ and $(\overset{2}{x}_o,x_1^{n-1}) \in R$, (iii) implies

$$(\overset{2}{x}_o,x_1^{n-2},x_o) \in R \text{ i.e. } (\overset{3}{x}_o,x_1^{n-2}) \in R.$$

Continuing this procedure, we finally get

(a) $(\overset{i}{x}_o,x_1^{n-(i-1)}) \in R$, for each $i \in \{2,\ldots,n+1\}$, and for arbitrary different $x_o,\ldots,x_{n-1} \in S$.

Applying the symmetry on (a), we get

$(x_1,\overset{i}{x}_o,x_2^{n+1-i}) \in R$, and $(\overset{i}{x}_o,x_2^{n+1-i},x_1) \in R$, and (iii) gives $(x_1,\overset{i}{x}_o,x_2^{n-i},x_1) \in R$ i.e. $(\overset{2}{x}_1,\overset{i}{x}_o,x_2^{n-i}) \in R$, and again, by (iii), $(\overset{3}{x}_1,\overset{i}{x}_o,x_2^{n-i-1}) \in R$.

Continuing, we get

$$(\overset{i}{x}_o,\overset{j}{x}_1,x_2^{n+2-i-j}) \in R, \quad i+j \leq n+1.$$

The same procedure gives

1) γ_{n+1}-and γ_n-symmetry produce the (whole) symmetry. Instead of γ_n one can take an arbitrary transposition.

$$(x_o^{i_o}, x_1^{i_1}, \dots, x_j^{i_j}, x_{j+1}^{n+j+1-i_o-\dots-i_j}) \in R \ , \ i_o+\dots+i_j \leq n+1 \ ,$$

completing by (ii), the proof of the reflexivity.

REMARK 7.

Note that the "only if" part of the proof of the preceding proposition shows that "some" of nA_1^*-transivity is included in reflexivity, in the case when there are equal element among $x_1,\dots,x_{n-1}$, for $(x_o^n) \in R$ and $(x_1^{n+1}) \in R$.

The fact that reflexivity does much more in generalized equivalences than in the binary case, can be shown by the following example, for n=2. Let R consists of all triples with equal coordinates (i.e. for $x \in S$, $(x,x,x) \in R$), and of arbitrary triples with different coordinates ($(x_1,x_2,x_3) \in R$, $x_i \neq x_j$, $i \neq j$), provided that R is $2\bar{A}_1$-transitive. Then no part of the symmetry can be produced, since there are no triples in R, being of the form (x,y,x), $x \neq y$.

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REZIME

TRANZITIVNE n-ARNE RELACIJE I
KARAKTERIZACIJE UOPŠTENIH EKVIVALENCIJA

Pickett |2| definiše uopštene relacije ekvivalencije i povezuje ih sa particijama tipa n koje je uveo Hartmanis |1|. U ovom radu dati su različiti tipovi uopštenih refleksivnih, simetričnih, kao i tranzitivnih relacija. Ispitane su osobine tih relacija i data su tvrdjenja koja ih povezuju. Najzad, dokazani su stavovi o različitim karakterizacijama uopštenih ekvivalencija.

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SOME PROPERTIES OF LINEAR OPERATORS OF DISCRETE FUNCTIONS

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Let $(P,+,\cdot)$ be a commutative ring with an identity element 1, and let $L \neq \emptyset$ be a finite set; and L^n its Cartesian product. Let us consider a set $F = \{f/f:L^n \to R\}$, where f is a discrete function. Some operators on L and F can be defined.

DEFINITION 1. $\phi_j : L \to L \quad , j \in I$

$$\partial : F \to F$$

where

$$\phi_k(\phi_j x) = \phi_k x \quad j,k \in I,\ x \in L$$

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\dots,x_{i-1},\phi_j x_i, x_{i+1},\dots,x_n) - f(x), \quad x \in L.$$

THEOREM 1. *All operators* ϕ_j, ϕ_i *, and* $\partial\ (j,i \in I)$ *and all discrete functions* f *and* g *satisfy the following properties:*

(1) $\partial(f_{\phi j})x_i = 0 \iff f = f(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$

(2) $\partial((f+g)_{\phi j})x_i = \partial(f_{\phi j})x_i + \partial(f_{\phi j})x_i \quad , \quad 1 \le i \le n$

(3) $\partial((kf)_{\phi j})x_i = k\partial(f_{\phi j})x_i, \quad k \in P,\ 1 \le i \le n$

(4) $\partial(\partial(f_{\phi j})x_i)_{\phi k})x_i = -\partial(f_{\phi j})x_i, \quad 1 \le i \le n$

(5) $\partial((f\cdot g)\phi_j)x_i = g\cdot\partial(f\phi_j)x_i + f\cdot\partial(g\phi_j)x_i +$

$$+ \partial(f\phi_j)x_i \cdot \partial(g\phi_j)x_i \ , \quad 1 \le i \le n$$

(6) $\partial((\partial f\phi_j)x_i)\phi_k)x_j = \partial((\partial f\phi_k)x_j)x_i \ , \quad i \neq j, \ 1 \le i \le n, \ 1 \le j \le n$

(7) $\partial(\partial(\ldots\partial(f\phi_j)x_i \ \ldots)\phi_j)x_i = (-1)^{m+1} \ \partial(f\phi_j)x_i, \quad 1 \le i \le n,$

∂ *is applied* m-*times*.

P r o o f. Relations (1), (3), (4) and (5) result immediately from Definition 1. Further, proving (2) let us use

$$(\phi_j\bar{x}_i) = (x_1,\ldots,x_{i-1},\phi_j x_i,\ldots,x_{i+1},\ldots,x_n)$$

thus

$$\partial((f+g)\phi_j)x_i = f(\phi_j\bar{x}_i)+g(\phi_j\bar{x}_i)-f(x)-g(x) =$$

$$= \partial(f\phi_j)x_i + \partial(g\phi_j)x_i \ , \quad 1 \le i \le n.$$

The proof of (5)

$$\partial((f\cdot g)\phi_j)x_i = f(\phi_j\bar{x}_i)\cdot g(\phi_j\bar{x}_i) - f(x)\cdot g(x) =$$

$$= f(x)\cdot\left[g(\phi_j\bar{x}_i)-g(x)\right]+g(x)\cdot\left[f(\phi_j\bar{x}_i)-f(x)\right] +$$

$$+ \left[f(\phi_j\bar{x}_i)-f(x)\right]\cdot\left[g(\phi_j\bar{x}_i)-g(x)\right] \quad .$$

Finally (7) will be proved by induction. For m=1 from (5) follows

$$\partial((\partial(f_{\phi_j})x_i)\phi_j)x_i = -\partial(f_{\phi_j})x_i \ .$$

This equality is true according to (4).

Let us suppose that (7) is satisfied for each n. Now, using Definition 1. again in (7) we shall get

$$\partial(((-1)^{m+1}\partial(f_{\phi_j})x_i)\phi_j)x_i = (-1)^{m+2} \ \partial(f_{\phi_j})x_i \ .$$

And so relation (7) is proved.

THEOREM 2. *For all discrete functions* f *and all operators* ϕ_j $(j \in I)$ *the following equality is held*

$$\partial((\ldots\partial(f_{\phi_{j_1}})_{x_{i_1}}\ldots)_{\phi_{j_m}})_{x_{i_m}} =$$

$$(8) \quad = \sum_{k=1}^{m} (-1)^{m-k} \sum_{i_1,\ldots,i_k}^{j_1,\ldots,j_m} f((\phi_{j_1}\bar{x}_{i_1}),\ldots,(\phi_{j_k}\bar{x}_{i_k})) + (-1)^m f(x),$$

where $\{i_1,\ldots,i_k\}$ *is a subset of* $\{j_1,\ldots,j_m\}$, $1 \le m \le n$.

P r o o f. For m=1 equality (8) becomes

$$\partial(f_{\phi_{j_1}})_{x_{i_1}} = f(\phi_{j_1}\bar{x}_{i_1}) - f(x) ,$$

it is true according to Definition 1.

Assume that equality (8) is true for every $m (m < n)$. Applying Definition 1. on (7) we get

$$\partial((\partial((\ldots\ \partial(f_{\phi_{j_1}})_{x_{i_1}}\ldots)_{\phi_{j_m}})_{x_{im}})_{\phi_{j_{m+1}}})_{x_{i_{m+1}}} =$$

$$= \sum_{k=1}^{m+1} (-1)^{m+1-k} \sum_{i_1,\ldots,i_k}^{j_1,\ldots,j_{m+1}} f((\phi_{j_i}\bar{x}_{i_1}),\ldots,(\phi_{i_k}\bar{x}_{i_k})) +$$

$$+ (-1)^{m+1} f(x) ,$$

thus, the theorem is proved.

THEOREM 3. *If*

$$f(\phi_j^n(x_n)) = f(\phi_j x_1,\ldots,\phi_{j_n} x_n)$$

$$f(\phi_i^n(x_n)) = f(\phi_i x_1,\ldots,\phi_{i_n} x_n) ,$$

then for all operators ϕ_j, ϕ_i, $i,j \in I$ *and all discrete functions the following equality holds:*

$$(9) \quad f(\phi_j^n(x_n) - f(\phi_i^n(x_n) = \sum_{m=1}^{n} \sum_{i_1,\ldots,i_m}^{1,\ldots,n} (\partial(\ldots\partial(f\,\phi_{j_1}(\phi_i x_1)_{x_{i_1}}\ldots$$

$$\ldots)_{\phi_{j_m}})_{x_{i_m}}) \quad (i_1 < i_2 < \ldots < i_m).$$

P r o o f. For m=1 (8) becomes

$$(10) \quad f(\phi_j^1 x_1) - f(\phi_i^1 x_1) = \partial(f_{\phi_j}(\phi_1 x_1))_{x_1}$$

and it is true according to Definition 1.

Let us suppose that (9) is true for n-1, i.e.

$$(11) \quad f(\phi_j^{n-1}(x_{n-1}), x_n) - f(\phi_i^{n-1}(x_{n-1}), x_n) = \partial(f_{\phi_j}(x_n))_{x_{n-1}}$$

Applying Definition 1. on (11) it will be transformed into equality (9).

Thus theorem 3. is proved.

In the next examples we shall show that these linear operators, given by Definition 1., cover partial derivatives of pseudo-Boolean functions and some parts of the following operators: partial derivatives of Boolean functions, Newton differences and lattice derivatives of discrete functions.

EXAMPLE 1. Let $f : L^n \to P$ be a generalized pseudo-Boolean function, and operators ϕ_j , $j \in I$

$$\phi_j x_i = a \ , \ j \in I \ , \ a \in L \ ,$$

where x_i, $1 \leq i \leq n$ are variables of generalized pseudo-Boolean functions. Operators ∂ are generalized pseudo-Boolean functions.

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) - f(x)$$

These operators ∂ are partial derivatives of generalized pseudo-Boolean functions with respect to the variables x_i, $1 \leq i \leq n$ (see |2|).

EXAMPLE 2. A binary operation $\oplus$ with the following properties is defined on L: for every a, b, c $\in$ L

$$a \oplus b = b \oplus a$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$
$$(\exists e \in L) \ e \oplus a = a \oplus e = a$$
$$(\forall a \in L)(\exists a' \in L) \ a \oplus a' = a' \oplus a = e$$
$$a \oplus a = a.$$

$L \neq \emptyset$ is a finite set, R is the set of real numbers, $f : L^n \to R$ is a real function. If the operators ϕ_j, $j \in I$ are defined in the following way

$$\phi_j x_i = x_i + h \ , \quad h \in L,$$

then operators ∂ are Newton differences

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\ldots,x_{i-1},x_2 \oplus h,x_{i+1},\ldots,x_n)-f(x).$$

EXAMPLE 3. Let (P,V,Λ) be a distributive lattice, $L \neq \emptyset$ a finite subset of P, and $f : L^n \to P$ a discrete function. If operators ϕ_j, $j \in I$ are defined

$$\phi_j x_i = x_i \vee a, \ a \in L,$$

where x_i, $1 \le i \le n$ are variables of discrete functions, then operators ∂ are discrete functions on the lattice

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\ldots,x_{i-1},x_i \vee a,x_{i+1},\ldots,x_n) \vee f(x) \ ,$$

$$1 \le i \le n$$

(see |3|).

EXAMPLE 4. Let $(P,+,\cdot)$ be a commutative ring with identity element 1. $L \neq \emptyset$ a finite subset of P, and $f : L^n \to P$. If operator ϕ_j, $j \in I$ are given by

$$\phi_j x_i = x_i + 1 \quad (\text{where } 1 + 1 = 1)$$

x_i, $1 \le i \le n$ are variables of f, then operators ∂ are

$$\partial(f_{\phi_j}(x))_{x_i} = f(x_1,\ldots,x_{i-1},x_i+1,x_{i+1},\ldots,x_n) - f(x)$$

(see |3|).

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REZIME

NEKA SVOJSTVA LINEARNIH OPERATORA DISKRETNIH FUNKCIJA

U ovom radu data je jedna nova definicija linearnih operatora koji pokrivaju parcijalne izvode generalisanih pseudo-Bulovih funkcija i delove sledećih operatora: parcijalne izvode Bulovih funkcija, neke Njutnove razlike, kao i neke latisne izvode diskretnih funkcija.

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A NOTE ON GENERALIZED PSEUDO-BOOLEAN FUNCTIONAL EQUATIONS WITH CONSTANT COEFFICIENTS AND n VARIABLES

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Let $L \neq \emptyset$ be a finite set; L^n its direct product, and let $(P, \oplus, \cdot, I)$ be a commutative ring, and I its identity. A function $f : L^n \to P$ is called a generalized pseudo-Boolean function.

DEFINITION 1. *The partial derivative of a generalized pseudo-Boolean function* $f : L^n \to P$ *with respect to variable* x_i $(1 \leq i \leq n)$ *is the generalized pseudo-Boolen function*

$$\frac{\partial f_a}{\partial x_i} : L^n \to P, \quad a \in L, \quad (1 \leq i \leq n)$$

where

$$\frac{\partial f_a}{\partial x_i}(x) = f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) - f(x)$$

$a \in L, \quad x = (x_1, \ldots, x_n), \quad (1 \leq i \leq n)$.

The following notations will be used:

$m(\alpha) = \alpha_1 + \alpha_2 + \ldots + \alpha_n, \ \alpha_i \in \{0,1\}$ (+ ordinary addition in the set of real numbers)

$M_n = \{\alpha \mid \alpha = \alpha_1\alpha_2 \ldots \alpha_n, \ \alpha_i \in \{0,1\}, \ m(\alpha) \geq 1\}$

$k(M_n) = 2^n - 1$

$\alpha_i \dot{+} \alpha_j, \ \alpha_i, \alpha_j \in \{0,1\}$ ($\dot{+}$ addition mod. 2)

$\partial x^\alpha = \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}, \ \alpha \in M_n$

$$b_i^{[o]} = 1, \quad b_i^{[1]} = b_i \ , \ i=1,\ldots,n, \quad b_i \in L$$

$$1 \cdot b_2 = b_i \cdot 1 = b_i$$

$$\partial^o x_i = 1$$

$$\frac{\partial^o f_{b_i}}{\partial x_i^o} = 1, \quad \frac{\partial^2 f_{b_1 b_2}}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \left(\frac{\partial f_{b_2}}{\partial x_j} \right)_{b_1}$$

$$P_i(\bar{a}_k,\bar{a}_j) = P_i(x_1,\ldots,x_{k-1},a_k,x_{k+1},\ldots,x_{j-1},a_j,x_{j+1},\ldots,x_n)$$

The following theorem has been proved (see |1|).

THEOREM 1. *The system of pseudo-Boolean functional equations*

(1) $$\frac{\partial f_{a_i}}{\partial x_i} = P_i(x), \quad i=1,\ldots,n \ ,$$

has a solution if and only if

(2.i) $$P_i(x_1,\ldots,x_{i-1},a,x_{i+1},\ldots,x_n) = 0, \quad i=1,\ldots,n \ ,$$

(2.ij) $$\frac{\partial P_{ja_i}}{\partial x_i} = \frac{\partial P_{ia_j}}{\partial x_j} \ , \quad i \neq j, \quad i,j=1,\ldots,n.$$

If conditions (2.i) and (2.ij) are fulfilles all the functions f *are determined by the formula*

$(3.i_1,\ldots,i_n)$ $$f(x) = c - \sum_{k=1}^{n} \oplus \ P_{i_k} \ (\bar{a}_{i_{k+1}},\ldots,\bar{a}_{i_n}) \oplus P_{i_n}(x) \ ,$$

where $i_1 i_2 \ldots i_n$ *are permutations of set* $\{1,2,\ldots,n\}$ *and* c *is a constant from* P.

The functional equation

(4) $$a \frac{\partial f_{b_1}}{\partial x} \oplus b \frac{\partial f_{b_2}}{\partial y} \oplus c \frac{\partial^2 f_{b_1 b_2}}{\partial x \partial y} = d\, g(x,y)$$

where $f : L^2 \to P$ is an unknown generalized pseudo-Boolean function $g : L^2 \to P$ is a known generalized pseudo-Boolen function ,

a,b,c,d are constants from, P, is obtained in |2| , where the next theorem is proved.

THEOREM 2. *The functional equation (4) has a solution if and only if*

(i) $d = ab(a \oplus b - c) \neq 0$

(ii) $$\frac{\partial^2 g_{b_1 b_2}}{\partial x \partial y} \oplus \frac{\partial g_{b_1}}{\partial x} \oplus \frac{\partial g_{b_2}}{\partial y} \oplus g = 0.$$

Here we shall observe a generalization of Theorem 2. Let us consider a generalized pseudo-Boolen functional equation with constant coefficients

$$(5) \qquad \sum_{\substack{\oplus \\ \alpha \in M_n}} a_\alpha \frac{\partial^{m(\alpha)} f_{b_1^{\lceil \alpha_1 \rceil} \cdots b_n^{\lceil \alpha_n \rceil}}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = d\,R(x) ,$$

where $f : L^n \to P$ is a unknown generalized pseudo-Boolen function $R : L^n \to P$ is a known generalized function, and constants a_α , $(\alpha \in M_n)$ d are from P.

For every generalized pseudo-Boolen function with constant coefficients and n variables the following property can be proved.

LEMMA 1. *If the left-hand side of the equation contains only the unknown function* f *and its partial derivatives , it can be written in the form of the left-hand side of (5).*

The proof follows directly from the properties of partial derivatives.

$$\frac{\partial^2 f_{ab}}{\partial x_i \partial y_j} = \frac{\partial^2 f_{ba}}{\partial x_j \partial x_i} \quad , \quad i \neq j, \quad (i,j=1,\dots,n)$$

$$\frac{\partial^m f_{a_1 \dots a_m}}{\partial x_i^m} = (-1)^{m+1} \frac{\partial f_{a_m}}{\partial x_i} \quad , \quad (1 \leq i \leq n)$$

$$\frac{\partial^m f_{a_1 \dots a_{k_1} \dots a_{k_2} \dots a_{k_p}}}{\partial x_{i_1}^{k_1} \dots \partial x_{i_p}^{k_p}} = (-1)^{m+p} \frac{\partial^p f_{a_{k_1} \dots a_{k_p}}}{\partial x_{i_1} \dots \partial x_{i_p}}$$

If n=2, equation (5) has the form (4), i.e.

$$a_{10} \frac{\partial f_{b_1}}{\partial x_1} \oplus a_{01} \frac{\partial f_{b_2}}{\partial x_2} \oplus a_{11} \frac{\partial^2 f_{b_1 b_2}}{\partial x_1 \partial x_2} = d\, P(x_1, x_2) \ .$$

If n=2, equation (5) has the form

$$a_{100} \frac{\partial f_{b_1}}{\partial x_1} \oplus a_{010} \frac{\partial f_{b_2}}{\partial x_2} \oplus a_{001} \frac{\partial f_{b_3}}{\partial x_3} \oplus$$

$$\oplus \ a_{110} \frac{\partial^2 f_{b_1 b_2}}{\partial x_1 \partial x_2} \oplus a_{101} \frac{\partial^2 f_{b_1 b_3}}{\partial x_1 \partial x_3} \ \oplus$$

$$\oplus \ a_{011} \frac{\partial^2 f_{b_2 b_3}}{\partial x_2 \partial x_3} \oplus a_{111} \frac{\partial^3 f_{b_1 b_2 b_3}}{\partial x_1 \partial x_2 \partial x_3} = d\, R(x_1, x_2, x_3) \ .$$

THEOREM 3. *Functional equation (5) has a solution if and only if*

$$(6.n) \qquad d = \prod_{m(\alpha)=1} a_\alpha \prod_{k=2}^{n} \Big(\prod_{i=1}^{\binom{n}{n-k}} (a_{\alpha_1^i} \oplus \dots \oplus a_{\alpha_k^i} -$$

$$- a_{\alpha_1^i \dot{+} \alpha_2^i} - a_{\alpha_1^i \dot{+} \alpha_3^i} - \dots - a_{\alpha_{k-1}^i \dot{+} \alpha_k^i} \oplus$$

$$\oplus \ a_{\alpha_1^i \dot{+} \alpha_2^i \dot{+} \alpha_3^i} \oplus a_{\alpha_1^i \dot{+} \alpha_2^i \dot{+} \alpha_4^i} \oplus \dots \oplus a_{\alpha_{k-2}^i \dot{+} \alpha_{k-1}^i \dot{+} \alpha_k^i} \oplus$$

$$\oplus \dots \oplus (-1)^{k+1} a_{\alpha_1^i \dot{+} \dots \dot{+} \alpha_k^i})\Big) \neq 0 \ ,$$

where $m(\alpha_1^i) = \dots = m(\alpha_k^i) = 1, \ \alpha_j^i \in M_n$,

and

$$(7.n) \qquad \sum_{\alpha\in M_n}^{\oplus} \frac{\partial^{m(\alpha)} R_{b_1^{\bar{\alpha}_1}\ldots b_n^{\bar{\alpha}_n}}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} \oplus R = 0 .$$

P r o o f. The partial derivatives

$$\frac{\partial^{m(p)} F_{b_1\ldots b_n}}{\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}}, \quad \beta=\beta_1\beta_2,\ldots,\beta_n \in M_n\setminus\{11\ldots1\},$$

of functional equation (5) from a system of functional equations

$$\sum_{\alpha\in M_n}^{\oplus} a_\alpha \frac{\partial^{m(\alpha)+m(\beta)} f_{b_1^{\beta_1}\ldots b_n^{\beta_n}}}{\partial x_1^{\alpha_1+\beta_1}\ldots\partial x_n^{\alpha_n+\beta_n}} = d \quad \frac{\partial^{m(\beta)} R_{b_1\ldots b_n}}{\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}}$$

$\beta\in M_n\setminus\{11\ldots1\}$.

System (8) has unique solution

$$\frac{\partial f_{b_1}}{\partial x_1},\ldots,\frac{\partial f_{b_n}}{\partial x_n} \qquad \text{if and only if the rank}$$

of the augmented matrix A_n' of the system (8) is 2^n-1.

For n=2 the augmented matrix of system (8) is equivalent to the matrix

$$A_2' = \left[\begin{array}{cccc} a_{10} & a_{01} & a_{11} & R \\ 0 & a_{01} & a_{01} & R\oplus\frac{\partial R_{b_1}}{\partial x_1} \\ 0 & 0 & a_{01}\oplus a_{10}-a_{11} & R\oplus\frac{\partial R_{b_1}}{\partial x_1}\oplus\frac{\partial R_{b_2}}{\partial x_2} \\ 0 & 0 & 0 & R\oplus\frac{\partial R_{b_1}}{\partial x_1}\oplus\frac{\partial R_{b_2}}{\partial x_2}\oplus\frac{\partial^2 R_{b_1b_2}}{\partial x_1\partial x_2} \end{array}\right].$$

Thus, rank $A_2' = 3$ if and only if

$$a_{10}\, a_{01}(a_{01} \oplus a_{01} - a_{11}) \neq 0$$

$$R \oplus \frac{\partial R_{b_1}}{\partial x_1} \oplus \frac{\partial R_{b_2}}{\partial x_2} \oplus \frac{\partial^2 R_{b_1 b_2}}{\partial x_1 \partial x_2} = 0 .$$

This is proved in Theorem 1.

According to (6.n) for n=2 it follows

$$(6.2') \qquad \prod_{m(\alpha)=1} a_\alpha \prod_{k=1}^{2} \Big(\prod_{i=1}^{\binom{2}{2-k}} (a_{\alpha_1^i} \oplus a_{\alpha_2^i} - a_{\alpha_1^i \dot{+} \alpha_2^i}) \Big) \neq 0 .$$

For n=3 the rank of the augmented matrix A_n' is 2^3-1 if and only if

$$\prod_{m(\alpha)=1} a_\alpha \prod_{k=2}^{3} \Big(\prod_{i=1}^{\binom{3}{3-k}} (a_{\alpha_1^i} \oplus a_{\alpha_2^i} \oplus \ldots \oplus a_{\alpha_k^i} -$$

$$(7.3') \qquad - a_{\alpha_1^i \dot{+} \alpha_2^i} - \ldots - a_{\alpha_{k-1}^i \dot{+} \alpha_k^i} \oplus a_{\alpha_1^i \dot{+} \alpha_2^i \dot{+} \alpha_3^i} \oplus$$

$$\oplus \ldots \oplus a_{\alpha_{k-2}^i \dot{+} \alpha_{k-1}^i \dot{+} \alpha_k^i} \oplus \ldots \oplus (-1)^{k+1} a_{\alpha_1^i \dot{+} \ldots \dot{+} \alpha_k^i})\Big) =$$

$$= \prod_{m(\alpha)=1} (a_{\alpha_i^1} \oplus a_{\alpha_2^1} - a_{\alpha_1^1 \dot{+} \alpha_2^1})(a_{\alpha_1^2} \oplus a_{\alpha_2^2} - a_{\alpha_1^2 \dot{+} \alpha_2^2})$$

$$(a_{\alpha_1^3} \oplus a_{\alpha_2^3} - a_{\alpha_1^3 \dot{+} \alpha_2^3})(a_{\alpha_1^1} \oplus a_{\alpha_2^1} \oplus a_{\alpha_3^1} - a_{\alpha_1^1 \dot{+} \alpha_2^1} -$$

$$- a_{\alpha_1^1 \dot{+} \alpha_3^1} - a_{\alpha_2^1 \dot{+} \alpha_3^1} \oplus a_{\alpha_1^1 \dot{+} \alpha_2^1 \dot{+} \alpha_3^1}) \neq 0,$$

$$m(\alpha_1^1) = m(\alpha_2^1) = m(\alpha_3^1) = 1.$$

According to (6.2′) and (7.3′) mathematical induction leads us to the proof of (6.n). (7.n) is proved in the same way according to (7.2) and (7.3). Thus, the theorem is proved.

REMARK. A new system of functional equations can be formed from the system of functional equations if it satisfies conditions (6.n) and (7.n)

$$(9) \qquad \frac{\partial f_{b_i}}{\partial x_i} = P_i(x) \;, \; i=1,\dots,n.$$

System (9) has a solution if and only if conditions (2.i) and (2.ii) are fulfilled.

(2.i) and (2.ii) follows immediately from (6.n) and (7.n).

This, the system of functional equations (9) has a solution which is determined by the formula $(3.1_i \dots i_n)$.

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REZIME

GENERALISANE PSEUDO-BULOVE FUNKCIONALNE JEDNAČINE
SA KONSTANTNIM KOEFICIJENTIMA I SA n PROMENLJIVIH

U ovom radu dati su potrebni uslovi (6.n) i (7.n) da generalisana funkcionalna jednačina (5) sa konstantnim koeficijentima i sa n promenljivih ima rešenja.

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SEMIGROUPS IN WHICH SOME BI-IDEAL IS A GROUP

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Semigroups containing minimal ideals are considered by A.H.CLIFFORD, |2|. If semigroups S contains at least one minimal left and at least one minimal right ideal, then it has a completely simple kernel or equivalently it has a quasi-ideal which is a group (see Theorem 3.2.|2| and Theorem 5.14. |9|). The structural theorem of this class is given in |7| and |8|. Here we will characterize this class using the notions of bi-ideal and AB-ideal (Theorem 1.). Using Theorem 1 we give a characterization of a semigroup in which some quasi-ideal is a special power joined semigroup. At the end we give a characterization of a semigroup in which every proper subsemigroup is a special power joined semigroup (Theorem 3.).

For nondefinied notions we refer to |5| and |9|.

A nonempty subset G of a semigroup S is called a left-A-ideal (right-A-ideal) of S if $sG \cap G \neq \emptyset$ $(Gs \cap G \neq \emptyset)$ for any $s \in S$. This notion is introduced by O.GROŠEK and L.SATKO, |4|. In this note introduce the concept of almost bi-ideal (AB-ideal).

DEFINITION 1. *A nonempty subset* B *of a semigroup* S *is called an almost bi-ideal (AB-ideal) of* S *if* $BsB \cap B \neq \emptyset$ *for every* $s \in S$.

If a left (right) A-ideal is a semigroup, then it is an AB-ideal.

If B is an AB-ideal of a semigroup S and $B \subset C \subset S$, then C is an AB-ideal of S.

The union of two AB-ideals of a semigroup S is also an AB-ideal of S.

The proof of the following proposition is obvious.

PROPOSITION 1. *Every nonempty subset of a semigroup S is an AB-ideal of S if and only if S is a rectangular band.*

PROPOSITION 2. *A semigroup S has a proper AB-ideal if and only if there exists an element $a \in S$ such that $(S \setminus a)s(S \setminus a) \cap (S \setminus a) \neq \emptyset$ for every $s \in S$.*

P r o o f. If a semigroup S contains a proper AB-ideal B and $a \notin B$, then $B \subset S \setminus a$ and $S \setminus a$ is a proper AB-ideal of S, i.e. $(S \setminus a)s(S \setminus a) \cap (S \setminus a) \neq \emptyset$ for every $s \in S$.

The converse is obvious.

As the corollary of Proposition 2. we have

PROPOSITION 3. *A semigroup S has no proper AB-ideals if and only if for every $a \in S$ there exists $s \in S$ such that $(S \setminus a)s(S \setminus a) = a$.*

PROPOSITION 4. *Let B be an AB-ideal of a semigroup S. Then xBy is an AB-ideal of S for every $x, y \in S$.*

P r o o f. First we have $BysxB \cap B \neq \emptyset$ for any $x, y, s \in S$ and this implies $xBysxBy \cap xBy \neq \emptyset$, i.e. that xBy is an AB-ideal.

PROPOSITION 5. *If B is a subsemigroup of a semigroup S and a minimal AB-ideal of S, then B is a subgroup of S.*

P r o o f. Let B be a minimal AB-ideal of a semigroup S which a subsemigroup of S. Then by Proposition 4. b_1Bb_2 is an AB-ideal of S. Since B is a minimal AB-ideal we have $B = b_1Bb_2$ for every $b_1, b_2 \in B$. This implies that B is a subgroup of S.

The following lemma is known.

LEMMA 1. |7|. *Let* B *be a bi-ideal of a semigroup* S. *Then* B *is a minimal bi-ideal of* S *if and only if* B *is a group.*

PROPOSITION 6. *A group* G *is an AB-ideal of a semigroup* S *if and only if* G *is a minimal bi-ideal of* S.

P r o o f. Let G be an AB-ideal of S. Then for every $s \in S$ there exist $g_1, g_2 \in G$ such that $g_1 s g_2 \in G$ and $g_1^{-1} g_1 s g_2 g_2^{-1} \in G$, i.e. $ese \in G$, where e is the identity of G. If follows from this that $gsh \in G$ for every $g, h \in G$. Hence, G is a bi-ideal of S. By Lemma 1. we have that G is a minimal bi-ideal of S.

The converse follows immediately.

LEMMA 2. |7|. *The union of all minimal bi-ideals of a semigroup* S *is an ideal of* S.

THEOREM 1. *Let be a semigroup. Then the following conditions are equivalent:*

(i) S *has Ab-ideal which is a group;*

(ii) S *has a bi-ideal which is a group;*

(iii) S *has a quasi-ideal which is a group;*

(iv) S *contains a completely simple kernel.*

P r o o f. (i) ⇒ (ii). This implication follows by Lemma 1. and by Proposition 6. (ii) ⇒ (iii). Let S has a bi-ideal which is a group and let K be the union of all bi-ideals which are groups. Then by Lemmas 1. and 2. K is an ideal od S. As K is completely regular semigroup we have that every bi-ideal of K is a quasi-ideal (Corollary 3.3. |6|). By Theorem 5.3. |9| we have that S contains a quasi-ideal which is a group. (iii) ⇒ (iv). This implication follows by Theorem 5.14. |9|. (iv) ⇒ (i). If a semigroup S contains a completely simple kernel K, then the maximal subgroups of K are bi-ideals of S and the assertion follows by Proposition 6.

COROLLARY 1. *Let* S *be a semigroup. The following conditions are equivalent:*

(i) S *is the union of its minimal bi-ideal;*
(ii) S *is the union of its minimal quasi-ideal;*
(iii) S *is the union of its AB-ideals which are groups;*
(iv) S *is completely simple.*

COROLLARY 2. *Let* S *be a semigroup. Then some left ideal of a semigroup* S *is a group if and only if* S *contains a kernel which is a right group.*

DEFINITION 2. |1| S *is a special power joined semigroup* (s.p.j.) *if for every* $a, b \in S$ *there exists a* $n \in N$ *such that* $a^n = b^n$.

LEMMA 3. |1| . S *is a* s.p.j. *semigroup if and only if* S *is a nil-extension of a periodic group.*

THEOREM 2. *Let* S *be a semigroup. Then some quasi-ideal of* S *is a* s.p.j. *semigroup if and only if* S *contains a completely simple periodic kernel.*

P r o o f. If some quasi-ideal Q of a semigroup S is a s.p.j. semigroup, then by Lemma 3. Q is a nil-extension of a periodic group G, and so G is a quasi-ideal od S. Hence, by Theorem 1. we have that S contains a completely simple kernel.

The converse is trivial.

COROLLARY 3. *Let* S *be a semigroup. Then some left ideal of* S *is a periodic group* (s.p.j. *semigroup) if and only if* S *contains a kernel which is a periodic right group.*

THEOREM 3. *Let* S *be a semigroup. Then the following conditions are equivalent:*

(1) Every proper subsemigroup of S *contains exactly one idempotent;*

(2) S *satisfies one of the following conditions:*

(i) $|S| = 2$;

(ii) S *is* s.p.j.;

(3) Every proper subsemigroup of S *is* s.p.j. .

P r o o f. (1) ⇒ (2). Let S be a semigroup in which every proper subsemigroup has exactly one idempotent. It is clear that S is periodic. Let $|S| > 2$. If S has exactly one idempotent, then S is s.p.j. . If S has two or more than two idempotents, then $S = \langle e,f \rangle$, where e and f are distinct idempotents of S. If $ef = fe$, then $S = \{e,f,ef\}$ which is a contradiction. If $ef \neq fe$, then

$$S = \{e,f\} \cup \langle ef\rangle \cup \langle efe\rangle \cup \langle fe\rangle \cup \langle fef\rangle .$$

In this case $\langle ef\rangle \cup \langle efe\rangle$ is a proper subsemigroup of S with exactly one idempotent $(ef)^n$. If $\{e\} \cup \langle ef\rangle \cup \langle efe\rangle$ is a proper subsemigroup of S, then $e = (ef)^n$ and therefore $e = ef$. From this $S = \{e,f,fe\}$. As $\{f,fe\}$ is a subsemigroup of S we have that $f = fe$ So $|S| = 2$, which is a contradiction. If $S = \{e\} \cup \langle ef\rangle \cup \langle efe\rangle$, then $f = (ef)^n$ and so $ef = f$. From this $S = \{e,f,fe\}$. As $\{f,fe\}$ is a subsemigroup of S we have that $f = fe$, which is not possible.

(2) ⇒ (3) ⇒ (1). These two implications follow immediately.

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REZIME

POLUGRUPE U KOJIMA NEKI BI-IDEAL JESTE GRUPA

U ovom radu daju se nove karakterizacije (pomoću bi-ideala i skoro bi-ideala (AB-ideala) za polugrupe koje sadrže potpuno prosto jezgro. Na osnovu ovog rezultata opisuju se polugrupe u kojima neki kvazi-ideal jeste specijalno stepeno povezana polugrupa. Na kraju opisuju se polugrupe u kojima svaka prava podpolugrupa sadrži tačno jedan idempotent.

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FUZZY GENERALIZED EQUIVALENCE RELATIONS AND PARTITIONS*)

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1. In |1| , Pickett defined a generalized equivalence relation on a set X containing at least n elements, as the (n+1)-ary relation E_n on X, satisfying the following three conditions, where S_{n+1} denotes a set of permutations on $\{1,\dots,n+1\}$:

E_{1n} : for all $x_1,\dots,x_n,(x_1,x_2,\dots,x_n,x_1) \in E_n$;

E_{2n} : for all $x_1,\dots,x_{n+1}$ and for each permutation $s \in S_{n+1}$,

if $(x_1,x_2,\dots,x_{n+1}) \in E_n$, then

$(x_{s(1)},x_{s(2)},\dots,x_{s(n+1)}) \in E_n$;

E_{3n} : for all $x_o,\dots,x_{n+1}$, $x_i \neq x_j$, for $i \neq j$, $i,j \in \{1,\dots,n\}$,

if $(x_o,\dots,x_n) \in E_n$ and $(x_1,\dots,x_{n+1}) \in E_n$ then

$(x_o,\dots,x_{n-1},x_{n+1}) \in E_n$

2. Fuzzy relations are discussed in |2| in the following way:

Let X be a set and $J([0,1],\wedge,\vee,\bar{\ },0,1)$ the distributive lattice on the unit interval $[0,1]$, where for $a,b \in [0,1]$,

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$$a \wedge b = \min(a,b), \quad a \vee b = \max(a,b), \quad \bar{a} = 1 - a.$$

3. The fuzzy n-ary relation R on x is defined by

$$\underline{R} \overset{def}{=} \{((x_1,\dots,x_n),\ m_{\underline{R}}(x_1,\dots,x_n)) \mid x_i \in X,\ i=1,\dots,n,\ m_{\underline{R}} : X^n \to [0,1]\}.$$

4. Hartmanis |3| defined a partition of type n on a set X with at least n elements, as a family P_n of subsets of X satisfying:

i) each $P \in P_n$ contains at least n elements; and

ii) each n different elements from X belong to exactly one $P \in P_n$.

As it is shown in |1|, each partition of type n determines one generalized equivalence relation on the same set, and vice versa, each (n+1)-ary equivalence determines one partition of type n.

In this article a fuzzy generalized equivalence relation will be defined and its properties will be described. In the second part, the notion of a fuzzy partition of type n will be given and it will be shown that there is a natural connection between fuzzy generalized equivalences and these partitions.

*

DEFINITION 1. *Let* X *be a set and* J *the distributive lattice given in* |2|. *A fuzzy generalized equivalence relation on* X *is* (n+1)*-ary, fuzzy relation* $\underline{R}_n$ *on* X, *satisfying the following conditions:*

$\underline{E}_{1n}$: *for all* $x_1,\dots,x_n$, $m_{\underline{R}_n}(x_1,\dots,x_n,x_1) = 1$

$\underline{E}_{2n}$: *for all* $x_1,\dots,x_{n+1}$, *and for each permutation* $s \in S_{n+1}$

$m_{\underline{R}_n}(x_1,\dots,x_{n+1}) = m_{\underline{R}_m}(x_{s(1)},\dots,x_{s(n+1)})$

$\underline{E}_{3n}$: *for all* $x_o,\dots,x_{n+1}$, *if* $m_{\underline{R}_n}(x_o,\dots,x_n) = a$ *and*

$m_{\underline{R}_n}(x_1,\dots,x_{n+1}) = b$ *then* $m_{\underline{R}_n}(x_o,\dots,x_{n-1},x_{n+1}) \geq a \wedge b$,

(a *and* b *are from* $[0,1]$, *and* $x_i \neq x_j$ *for* $i \neq j$, $i,j \in \{1,\dots,n\}$)

The following two propositions are the generalizations of the Theorems of decomposition and synthesis for fuzzy relations (|2|), connecting the fuzzy generalized equivalences and (n+1)-ary equivalence relations given in |1|.

PROPOSITION 1. *Let* $\underline{R}_n$ *be a fuzzy generalized equivalence relation on* X. *Then*

$$\underline{R}_n = \bigcup_{a\in[0,1]} a\cdot R_a,$$

where

$$a_1 \le a_2 \quad \textit{implies} \quad R_{a_2} \le R_{a_1}$$

R_a $(a \in [0,1])$ *is the generalized equivalence relation in the sense of* |1|.

P r o o f. Let

$$R_a \overset{\text{def}}{=} \{(x_1,\dots,x_{n+1}) \mid x_i \in E,\ i=1,\dots,n,$$
$$m_{\underline{R}_n}(x_1,\dots,x_{n+1}) \ge a,\ a \in [0,1]\}.$$

It is obvious that

$$a_i \le a_j \quad \text{implies} \quad R_{a_j} \le R_{a_i}.$$

R_a, thus, is not a fuzzy relation.

Let now for a and b from $[0,1]$

$$m_b \cdot R_a(x_1,\dots,x_{n+1}) \overset{\text{def}}{=} b \wedge m_{R_a}(x_1,\dots,x_{n+1}).$$

Then,

$$m_{\bigcup_{a\in[0,1]}} a \cdot R_a(x_1,\dots,x_{n+1}) = \bigvee_a a \cdot m_{R_a}(x_1,\dots,x_{n+1}) =$$

$$\bigvee_{a \le m_{\underline{R}_n}} (x_1,\dots,x_{n+1}) = m_{\underline{R}_n}(x_1,\dots,x_{n+1}),$$

since

$$m_{R_a}(x_1,\dots,x_n) = \begin{cases} 1, & \text{for } a \le m_{\underline{R}_n}(x_1,\dots,x_{n+1}) \\ 0, & \text{otherwise.} \end{cases}$$

The decomposition of $\underline{R}_n$ into (n+1)-ary relations R_a is thus proved. Now we have to show that these relations are generalized equivalences.

Since $\underline{R}_n$ satisfies $\underline{E}_{n1}$, for all $x_1,\dots,x_n$ from X

$$m_{\underline{R}_n}(x_1,\dots,x_n,x_1) = 1,$$

then $(x_1,\dots,x_n,x_1)$ belongs to R_a for each a from $[0,1]$, i.e. R_a is reflexive (in the sense of E_{n1}) .

If $(x_1,\dots,x_{n+1})$ is in R_a, this means that $m_{\underline{R}_n}(x_1,\dots$ $\dots,x_{n+1}) \geq a$, and by symmetry (E_{3n}) for each $s \in S_{n+1}$

$$m_{\underline{R}_n}(x_{s(1)},\dots,x_{s(n+1)}) \geq a \text{ , and thus}$$

$$(x_{s(1)},\dots,x_{s(n+1)}) \in R_a \text{ , i.e. the condition } E_{n2} \text{ is}$$
satisfied.

Finally if $(x_o,\dots,x_n)$ and $(x_1,\dots,x_{n+1})$ both belong to R_a, then

$$m_{\underline{R}_n}(x_o,\dots,x_n) \geq a \quad \text{and } m_{\underline{R}_n}(x_1,\dots,x_{n+1}) \geq a \text{ ,}$$

and by E_{3n}

$$m_{\underline{R}_n}(x_o,\dots,x_{n-1},x_{n+1}) \geq a \text{ ,}$$

i.e. $\quad (x_o,\dots,x_{n-1},x_{n+1}) \in R_a$,

completing the proof.

It is obvious that the method used in the proof of the previous proposition can be applied in the opposite way also, i.e. that the converse is also true:

PROPOSITION 2. *If* $R_a (a \in [0,1])$ *are generalized equivalence relations such that*

$$a_i \leq a_j \quad \textit{implies} \quad R_{a_j} \leq R_{a_i} \text{ ,}$$

then $\underline{R}_n$ *, defined by*

$$\underline{R}_n = \bigcup_{a \in [0,1]} a \cdot R_a$$

is a fuzzy generalized equivalence relation (in the sense of Definition 1).

* *

DEFINITION 2. *Let* $\underline{R}$ *be the fuzzy generalized equivalence relation on* X. *The equivalence class of the level* p ($p \in [0,1]$) *denote by* $|\{x_1^n\}|_p$ *) , *for* $x_1,\ldots,x_n \in X$ *is defined by*

$$|\{x_1^n\}|_p = \{y \mid m_{\underline{R}}(x_1,\ldots,x_n,y) \geq p\}.$$

DEFINITION 3. *Fof* $p \in [0,1]$, *the quotient-set of the level* p *for* $\underline{R}$ *on* X, *denoted by* $(X/\underline{R})_p$, *is*

$$(X/\underline{R})_p = \{|\{x_1^n\}|_p \mid x_1,\ldots,x_n \in X\} .$$

DEFINITION 4. *The fuzzy quotient-set* $X/\underline{R}$ *is a fuzzy set on* $\bigcup_{p\in[0,1]} (X/\underline{R})_p$, *such that*

$$m_{(X/\underline{R})}(|\{x_1^n\}|_p) = \sup\,(\rho \mid |\{x_1^n\}|_\rho = |\{x_1^n\}|_p).$$

PROPOSITION 3. *If* $\underline{R}$ *is a fuzzy generalized equivalence relation on* X , *then* $(X/\underline{R})_p$, $p \in [0,1]$, *is a partition of type* n , *in the sense of 4.*

P r o o f. By Proposition 1., R_p is a generalized equivalence relation. From that, and the fact that

$$|\{x_1^n\}|_p = \{y \mid m_{\underline{R}}(x_1,\ldots,x_n,y) \geq p\} = \{y \mid (x_1,\ldots,x_n,y) \in R_p\}$$

it follows that conditions (i) and (ii), 4., are satisfied.

DEFINITION 5. *Let* X *be a set containing at least* n *elements. A fuzzy partition of type* n *on* X *is a fuzzy set* $\underline{\Pi}(X)$ *on* $P(X)$, *satisfying:*

a) $\{P \mid P \in P(x) \qquad m_{\underline{\Pi}(x)}(P) = 1\}$

is a partition of type n *on* X.

b) Let $P,Q \in P(x)$, $m_{\underline{\Pi}(x)}(P) \neq 0$, $m_{\underline{\Pi}(x)}(Q) \neq 0$. *Now, if*

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*) $|\{x_1^n\}|_p$ stands for $|\{x_1,\ldots,x_n\}|_p$

card $(P \cap Q) \geq n$ *then*

$$m_{\underline{\Pi}(x)}(P) \leq m_{\underline{\Pi}(x)}(Q) \qquad \textit{iff} \quad Q \subset P$$

(where the equality holds on the left side iff it holds on the right).

The following proposition is the direct consequence of Definitions 4. and 5.

PROPOSITION 4. *Let* $\underline{R}$ *be the fuzzy generalized equivalence relation on* X. *Then* $X/\underline{R}$ *is a fuzzy partition of type* n *on* X.

PROPOSITION 5. *Let* $\underline{\Pi(X)}$ *be the fuzzy partition of type* n *on* X. *Then the fuzzy* (n+1)*-ary relation* $\underline{R}_\Pi$ *on* X *defined by*

$$(*) \qquad m_{\underline{R}_\Pi}(x_1,\dots,x_{n+1}) = \begin{cases} p, \text{ if there is } P \in P(x) \text{ such that} \\ x_1,\dots,x_{n+1} \in P \\ \quad \text{and} \\ m_{\underline{\Pi}(x)}(P) = p, \\ 0 \qquad \text{otherwise}, \end{cases}$$

is a fuzzy quivalence relation on X.

P r o o f. $\underline{R}_\Pi$ is reflexive: (a) implies that

$$m_{\underline{R}_\Pi}(x_1,\dots,x_n,x_1) = 1.$$

$\underline{R}_\Pi$ is symmetric: If $m_{\underline{R}_\Pi}(x_1,\dots,x_{n+1}) = p$, by (*) then there is $P \in P(x)$ such that $x_1,\dots,x_{n+1} \in P$ and $m_{\underline{\Pi}(x)}(P) = p$, which holds for each permutation of $x_1,\dots,x_{n+1}$.

$\underline{R}_\Pi$ is tranzitive: Let $m_{\underline{R}_\Pi}(x_o,\dots,x_n) = p$

$m_{\underline{R}_\Pi}(x_1,\dots,x_{n+1}) = \dot{q}$, $x_i \neq x_j$, for $i \neq j$, $i,j \in \{1,\dots,n\}$. Then there are P and Q in $P(X)$ such that

$$x_o,\dots,x_n \in P \text{ and } x_1,\dots,x_{n+1} \in Q.$$

First let $p < q$. Then by (b) $Q \subset P$ and $x_o, \ldots, x_{n+1} \in P$, i.e.

$$m_{\underline{R}_{\Pi}}(x_o, \ldots, x_{n-1}, x_{n+1}) = p \geq p \wedge q \ .$$

If $p = q$, then $P = Q$, by (b) (the part concerning the equality). Then again

$$x_o, x_1, \ldots, x_n, x_{n+1} \in P = Q \ , \quad \text{and thus}$$

$$m_{\underline{R}_{\Pi}}(x_o, \ldots, x_{n-1}, x_{n+1}) = p = q.$$

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REZIME

RASPLINUTE UOPŠTENE RELACIJE EKVIVALENCIJE
I PARTICIJE

U radu se definišu rasplinute uopštene relacije ekvivalencije i dokazuju stavovi o dekompoziciji i sintezi tih relacija, pomoću ekvivalencija definisanih u |1|. Daje se i pojam rasplinute particije tipa n (uopštenje pojma iz |3|), i pokazuje da postoji veza izmedju tih particija i rasplinutih generalisanih ekvivalencija, preko odgovarajućih rasplinutih faktor - skupova.

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STRUCTURE OF GENERALIZED EQUIVALENCES CONTAINED IN $(2,n\bar{A}_1)$ - RT RELATIONS

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It is well-known that if ρ is a binary reflexive and transitive relation on S, then $\sigma = \rho \cap \rho^{-1}$ is an equivalence on S, and that an ordering χ can be defined on S/σ by: $(X,Y) \in \chi$ iff $(x,y) \in \rho$ for any $x \in X$, $y \in Y$. Binary relation σ is a maximal (in regard to the set inclusion) equivalence relation contained in ρ , and morever, the set of all equivalences in ρ is a complete lattice.

The class of binary reflexive and transitive relations is uniquely determined. In |3| it is shown that this is not the case with (n+1)-ary relations, when $n \geq 2$. Here we consider 2-reflexive, $n\bar{A}_1$-transitive, (n+1)-ary relations on the given set S, denoted as $(2,n\bar{A}_1)$-RT relations, induced among some other classes of (n+1)-ary relations in |3|. The structure of generalized equivalences (defined in |1|) included in such an generalized quasi-order is the subject of this article. We show that this poset always has the maximal elements, and we give the necessary and sufficent conditions under which it is a complete lattice. Finally , we describe two generalized orderings induced on the corresponding partition of type n (Hartmanis, see |1|) by one class of $(2,n\bar{A}_1)$-RT relations. We note that the considerations of some of these problems, we started in |2|.

*

1. (n+1)-ary relation ρ on S is (i_1^t)-reflexive, $i_1,\ldots,i_t \in \{2,\ldots,n+1\}$, iff

$$(a_1^{i_1-1}, a, a_{i_1+1}^{i_2-1},\ldots,a_{i_{t-1}+1}^{i_t-1},a,a_{i_t+1}^{n+1}) \in \rho ,$$

for all $a_1,\ldots,a_{i_1-1},a_{i_1+1},\ldots,a_{i_t-1},a_{i_t+1},\ldots,a_{n+1},a \in S$ 1).

ρ is t-reflexive, $t \in \{2,\ldots,n+1\}$, iff it is (i_1^t)-reflexive for all different $i_1,\ldots,i_t \in \{1,\ldots,n+1\}$ 2). An (i_1^{n+1})-reflexive relation ρ is (trivialy) (n+1)-reflexive, and it is described by the formula:

$$(\forall a \in S)((\overset{n+1}{a}) \in \rho) .$$

2. (n+1)-ary relation ρ on S is k-antisymmetric, $k \in \{2,\ldots,n+1)\}$, iff for all $a_1,\ldots,a_k \in S$ the following is satisfied:

If all permutations of $a_1,\ldots,a_k$ are included in (n+1)-tuples of ρ, then $a_1 =\ldots= a_k$.

3. (n+1)-ary relation ρ on S is $n\bar{A}_1$ -transitive iff from $(a_o^n) \in \rho$, $(a_1^{n+1}) \in \rho$, and $a_i \neq a_j$ for $i\neq j$, $i,j \in \{1,\ldots,n\}$, it follows that $(a_o^{n-1},a_{n+1}) \in \rho$, for all $a_o,\ldots,a_{n+1} \in S$.

REMARK:

Some other generalizations of the antisymmetric and transitive relations are given in |3|,|4| and |5|.

4. (n+1)-ary relation ρ on S is symmetric, iff for all $a_1,\ldots,a_{n+1} \in S$, the following is satisfied:
$(a_1^{n+1}) \in \rho$ implies $(a_{\pi(1)},\ldots,a_{\pi(n+1)}) \in \rho$, for each $\pi \in \{1,\ldots,n+1\}!$.

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1) For t=2, this is (i,j)-reflexivity from |5|.

2) "Reflexive" in |5| is 2-reflexive here.

5. $(n+1)$-ary relation ρ on S is generalized equivalence, iff it is $(1,n+1)$-reflexive, symmetric, and $n\bar{A}_1$-transitive (|1|).

6. We denote by d_2 the intersection of all 2-reflexive $(n+1)$-ary relations on S, i.e.

$$d_2 = \{(a_0^n) \mid a_0,\dots,a_n \in S,\ a_i = a_j \text{ for some } i,j \in \{0,\dots,n\}\}.$$

(d_2 is $n\bar{A}_1$-transitive too, see |3|).

In the following, we assume that $|S| \geq n$.

* *

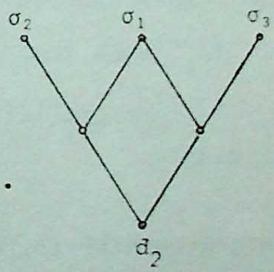
To illustrate the problems that arise in considering the structure of equvalences contained in generalized quasi-order , we start with one example.

EXAMPLE 1. $S = \{a,b,c,d,e\}$, $n=2$.

$$\rho = \pi(a,b,c)^{1)} \cup \pi(b,c,d) \cup \pi(b,c,e) \cup \pi(a,b,e) \cup \pi(a,c,e) \cup$$
$$\{(a,b,d),(d,b,a),(a,c,d),(d,c,a),(b,a,d),(b,d,a),$$
$$(c,a,d),(c,d,a),(e,a,d),(e,b,d),(e,c,d),(a,e,d),$$
$$(b,e,d),(c,e,d),(d,b,e),(d,c,e),(b,d,e),(c,d,e)\}.$$

ρ is $(2,2\bar{A}_1)$-RT relation on S. The following relations are maximal ternary equivalences contained in ρ .

$$\sigma_1 = d_2 \cup \pi(a,b,c) \cup \pi(b,c,e) \cup \pi(a,c,e) \cup \pi(a,b,e) ;$$
$$\sigma_2 = d_2 \cup \pi(b,c,d) \cup \pi(a,b,a);$$
$$\sigma_3 = d_2 \cup \pi(b,c,d) \cup \pi(a,c,e);$$



Hasse diagram of the partialy order set of equivalences in ρ illustrates the situation.

1) $\pi(a,b,c)$ denotes $\{(a,b,c),(a,c,b),(b,a,c),(b,c,a),(c,a,b),(c,b,a)\}$

THEOREM 1. *Let* ρ *be* $(2,n\bar{A}_1)$ *-RT relation on* S . *Now, if* E *denotes the set of all equivalence relations* σ *on* S *, such that* $\sigma \subseteq \rho$*, then the partially ordered set* $\langle E, \subseteq \rangle$ *contains at least one maximal equivalence relation.*

P r o o f. $E \neq \emptyset$, since $d_2 \subseteq E$ (see |3|). Let $\{\sigma_i; i \in I\}$ be a chain in $\langle E, \subseteq \rangle$. $\bar{\sigma} = \bigcup_{i \in I} \sigma_i$ is an upper bound for that chain. Realy, $\bar{\sigma}$ is 2-reflexive, since $d_2 \subseteq \bar{\sigma}$. $\bar{\sigma}$ is symmetric: if $(a_1^{n+1}) \in \bar{\sigma}$ then $(a_1^{n+1}) \in \sigma_i$, for some $i \in I$, and since σ_i is symmetric, $(a_{\pi(1)}, \ldots, a_{\pi(n+1)}) \in \sigma_i$, for every $\pi \in \{1,\ldots,n+1\}!$, and thus $(a_{\pi(1)}, \ldots, a_{\pi(n+1)}) \in \bar{\sigma}$, for every π. $\bar{\sigma}$ is $n\bar{A}_1$-transitive: suppose that $(a_0^n) \in \bar{\sigma}$ and $(a_1^{n+1}) \in \bar{\sigma}$, and $a_1, \ldots, a_n$ are different. Then $(a_0^n) \in \sigma_i$ and $(a_1^{n+1}) \in \sigma_j$, for some $i,j \in I$. Let $\sigma_i \subseteq \sigma_j$. Then both $(n+1)$-tuples belong to σ_j, and by $n\bar{A}_1$-transitivity it follows that $(a_0^{n-1}, a_{n+1}) \in \sigma_j$, and thus $(a_0^{n-1}, a_{n+1}) \in \bar{\sigma}$. By Zorn's Lemma we conclude that $\langle E, \subseteq \rangle$ has a maximal element.

Generalizing binary case for $(2,n\bar{A}_1)$-RT relation ρ on S, we get the following definition of the relation σ_ρ:

(1) $(a_1^{n+1}) \in \sigma_\rho$ iff $(a_{\pi(1)}, \ldots, a_{\pi(n+1)}) \in \rho$, for every $\pi \in \{1,\ldots,n+1\}!$.

It is obvious that the following proposition holds.

Lemma 2. If $\sigma \in E$, *then*

a) $(a_1^{n+1}) \in \sigma$ *implies* $(a_{\pi(1)}, \ldots, a_{\pi(n+1)}) \in \rho$, *for every* $\pi \in \{1,\ldots,n+1\}!$

b) $d_2 \subseteq \sigma \subseteq \sigma_\rho$.

•

Up to now we have found that for $n > 1$

i) σ_ρ is not always transitive;

ii) $\langle E, \subset \rangle$ can have more than one maximal equivalence; and

thus iii) $\langle E, \subset \rangle$ is not always a lattice.

If the following we discuss some of these problems.

THEOREM 3. $\sigma_\rho = \cup E$ *(union of all* (n+1)*-ary equivalences in* E *).*

P r o o f. 1) $\cup E \subset \sigma_\rho$. Realy, if $(a_1^{n+1}) \in \sigma$, $\sigma \in E$, then by a), Lemma 2, $(a_1^{n+1}) \in \sigma_\rho$.

2 $\sigma_\rho \subset \cup E$. Indeed, if $(a_1^{n+1}) \in \sigma_\rho$, then for every $\pi \in \{1, \ldots, n+1\}!$ $(a_{\pi(1)}, \ldots, a_{\pi(n+1)}) \in \sigma_\rho$, and (a_1^{n+1}) belongs at least to equivalence relation $\sigma = d_2 \cup \{(a_{\pi(1)}, \ldots, a_{\pi(n+1)}); \pi \in \{1, \ldots \ldots, n+1\}!\}$. Thus, $(a_1^{n+1}) \in \cup E$.

It follows from 1) and 2) that $\sigma_\rho = \cup E$.

It is obvious that $\langle E, \subseteq \rangle$ is a meet semilattice with zero d_2. Now we can give the necessary and sufficent conditions under which it is a lattice.

We start with the following definition of a special (n+1)-ary quasiorder.

(n+1)-ary relation ρ on S is $(2, n\bar{A}_1)^*$ RT relation iff it is $(2, n\bar{A}_1)$-RT relation and the following is satisfied:

(*) If

(a) $(a_{\alpha(0)}, \ldots, a_{\alpha(n)}) \in \rho$ and $(a_{\beta(1)}, \ldots, a_{\beta(n+1)}) \in \rho$,

for each $\alpha \in \{0, \ldots, n\}!$ and for each $\beta \in \{1, \ldots, n+1\}!$, and $a_0, \ldots \ldots, a_{n+1}$ are different elements of S, then, with each cosequence

(b) $(b_1, \ldots, b_{n+1}) \in \rho$, $(b_1, \ldots, b_{n+1} \in \{a_0, \ldots, a_{n+1}\})$,

for the corresponding premises of (a) by $n\bar{A}_1$-transitivity, in ρ is also

($\bar{b}$) $(b_1, \ldots, b_{n-1}, b_{n+1}, b_n)$,

for all $a_0, \ldots, a_{n+1} \in S$.

THEOREM 4. *If* ρ *is* $(2,n\bar{A}_1)^{\star}$ *RT ralation, then* $\mathsf{U}E$ *is* $(n+1)$*-ary equivalence relation on* S.

P r o o f.

a) $\mathsf{U}E$ is (by definition) 2-reflexive and symmetric.

b) $\mathsf{U}E$ ia $n\bar{A}_1$- transitive:

Let $(a_o^n) \in \mathsf{U}E$ and $(a_1^{n+1}) \in \mathsf{U}E$, $a_i \neq a_j$, for $i \neq j$, $i,j \in \{1,\ldots,n\}$. By $(\bar{1})$ this is equivalent to (a) in (*).

b_1) If $a_o,\ldots,a_{n+1}$ are not all different, and the conditions for the application of $n\bar{A}_1$-transitivity are satisfied, then $(a_{\gamma(o)},\ldots,a_{\gamma(n-1)}, a_{\gamma(n+1)}) \in \rho$, for each $\gamma \in \{0,\ldots,n-1,n+1\}!$, because of

1) 2-reflexivity of ρ ; or

2) the consequence becomes one of the premises in (a). Thus, $(a_o,\ldots,a_{n-1},a_{n+1}) \in \mathsf{U}E$.

b_2) Suppose now that $a_o,\ldots,a_{n+1}$ are all different. Then, starting with (a), we get that with $(a_o,\ldots,a_{n-1},a_{n+1})$,all $(n+1)$ tuples of the form

$$(a_o,a_{\eta(1)},\ldots,a_{\eta(n-1)},a_{n+1}), \quad \eta \in \{1,\ldots,n-1\}! \ .$$

also belong to ρ. Since 2-reflexive and $n\bar{A}_1$-transitive relation admits all cyclic permutations of first n coordinates of it's elements, and by (*), it follows that for each

$$\gamma \in \{0,\ldots,n-1,n+1\}!, \ (a_{\gamma(o)},\ldots,a_{\gamma(n-1)},a_{\gamma(n+1)}) \in \rho \ .$$

Thus, $(a_o,\ldots,a_{n-1},a_{n+1}) \in \mathsf{U}E$, completing the proof of the proposition.

•

THEOREM 5. $\langle E, \subset \rangle$ *is a complete lattice iff* ρ *is* $(2,n\bar{A}_1)^{\star}$ *RT relation.*

P r o o f.

a) Let ρ be $(2,n\bar{A}_1)^{\star}$-RT relation on S. Then by Theorem 4., $\mathcal{U}E \in E$. That is why $\mathcal{U}E$ is the only maximal element in $\langle E, \subset\rangle$. and clearly, the gratest one. E is closed under arbitrary intersections, and thus, it is a complete lattice.

b) Let now $\langle E, \subset\rangle$ be a complete lattice. Then it has a unit element $\mathcal{U}E$. $\mathcal{U}E$ is thus $(2,n\bar{A}_1)^{\star}$ RT relation. Indeed, let

(o) $\quad (a_o^n) \in \mathcal{U}E$ and $(a_1^{n+1}) \in \mathcal{U}E$ imply $(a_o^{n-1}, a_{n+1}) \in \mathcal{U}E$.

Then by $(\bar{1})$

$(a_o^n) \in \mathcal{U}E$ iff $(a_{\alpha(o)},\dots,a_{\alpha(n)}) \in \rho$, for every $\alpha \in \{0,\dots,n\}!$;

$(a_1^{n+1}) \in \mathcal{U}E$ iff $(a_{\beta(1)},\dots,a_{\beta(n+1)}) \in \rho$, for every $\beta \in \{1,\dots,n+1\}!$

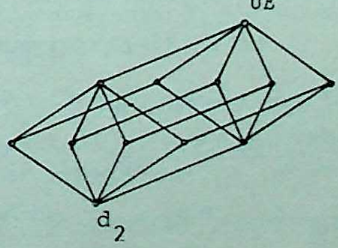
$(a_o^{n-1}, a_{n+1}) \in \mathcal{U}E$ iff $(a_{\gamma(o)},\dots,a_{\gamma(n-1)},a_{\gamma(n+1)}) \in \rho$, for every $\gamma \in \{0,\dots,n-1,n+1\}!$.

In this way it is shown that ρ satisfies (*), and thus it is $(2,n\bar{A}_1)^{\star}$-RT relation.

EXAMPLE 2. $S = \{a,b,c,d,e,f\}$, $n = 2$.

$$\rho = d_2 \cup \pi(a,b,c) \cup \pi(a,b,d) \cup \pi(a,c,d) \cup \pi(b,c,d) \cup \pi(d,e,f) \qquad \text{(i)}$$
$$\cup \{(a,b,e),(b,a,e),(a,c,e),(c,a,e),(a,b,e),(d,a,e),(b,c,e),$$
$$(c,b,e),(b,d,e),(d,b,e),(c,d,e),(d,c,e),(a,d,f),(b,a,f),$$
$$(a,c,f),(c,a,f),(a,d,f),(d,a,f),(b,c,f),(c,b,f),(b,d,f),$$
$$(d,b,f),(c,d,f),(d,c,f)\}.$$

ρ is $(2,n\bar{A}_1)^{\star}$-RT relation. The lattice $\langle E, \subset\rangle$ is given by it's Hasse diagram, where zero is d_2, and unit is $\mathcal{U}E$, described by (i) in ρ.



Since in binary case there is only one class of RT relations, and it satisfies (*), the fact that for n=1 $\langle E, \subseteq \rangle$ is a lattice is a direct consequence of Theorem 5.

* * *

Consider now the binary relation χ, defined at the beginning of the article, concerning the induced order on the partition. The following two theorems deal with the same problems for (n+1)-ary relations.

THEOREM 6. *Let* ρ *be* $(2, n\bar{A}_1)^{*}$-RT *relation on* S , *and denote* $\cup E$ *by* σ. *Let* S/σ *be the corresponding partition of type* n. *Now, if* χ *is* (n+1)-*ary relation on* S/σ, *defined by*

$$(x)\qquad (Q_1^{n+1}) \in \chi \text{ iff } (x_{i_1},\dots,x_{i_{n+1}}) \in \rho, \text{ for all}$$

$$(x_{i_1},\dots,x_{i_{n+1}}) \in Q_1 \times \dots \times Q_{n+1}, Q_1,\dots,Q_{n+1} \in S/\sigma ,$$

then in this way induced (by ρ*) relation* χ *is* (n+1)-*reflexive,* (n+1) -*antisymmetric, and* $n\bar{A}_1$-*transitive.* 1)

P r o o f.

a) $(\overset{n+1}{Q}) \in \chi$ iff $(x_{i_1},\dots,x_{i_{n+1}}) \in \rho$, for all

$x_{i_1},\dots,x_{i_{n+1}} \in Q$, and this is true since $(x_{i_1},\dots,x_{i_{n+1}}) \in$

$\in \sigma \subseteq \rho$.

Thus χ is (n+1)-reflexive.

b) χ is (n+1)-antisymmetric:

Let $(Q_{\pi(1)},\dots Q_{\pi(n+1)}) \in \chi$, for each $\pi \in \{1,\dots,n+1\}!$. Then

$(x_{\pi(i_1)},\dots,x_{\pi(i_{n+1})}) \in \rho$, whenever this (n+1)-tuple belongs

to $Q_{\pi(1)} \times \dots \times Q_{\pi(n+1)}$, i.e. when

1) These properties are consistent as shown in |3|.

$(x_{i_1},\dots,x_{i_{n+1}}) \in \sigma$. But this means that $x_{i_1},\dots,x_{i_{n+1}}$ belong to the same class, i.e. $Q_1 = \dots = Q_{n+1}$.

c) χ is $n\bar{A}_1$-transitive:

Let $(Q_o^n) \in \chi$, $(Q_1^{n+1}) \in \chi$, $Q_i \neq Q_j$, for $i \neq j$, $i,j \in \{1,\dots,n\}$. This holds if and only if

$(x_{i_o},\dots,x_{i_n}) \in \rho$, $(x_{i_1},\dots,x_{i_{n+1}}) \in \rho$, whenever $x_{i_j} \in Q_j$ $(j \in \{0,\dots,n+1\})$.

Then $(x_{i_o},\dots,x_{i_{n-1}},x_{i_{n+1}}) \in \rho$, $x_{i_j} \in Q_j$, $j \in \{0,\dots,n-1,n+1\}$,

since : a) ρ is 2-reflexive (if $x_{i_1},\dots,x_{i_n}$ are not all different) or b) ρ is $n\bar{A}_1$-transitive (otherwise).

By a), b) and c), the proof is complete.

Generalized ordering relation χ, defined in the preceding proposition, in binary case reduces to the usual one. The same is with the relation ψ, given in the following proposition. This one has already been defined in |3|, but with some unpreciseness included. That is why we repeat it here, together with one example, illustrating both, χ and ψ.

THEOREM 7. *Let* $|S| > n$, $n \neq 2$ *and* ρ, σ, *and* S/σ *be as in Theorem 6. Define* $(n+1)$*-ary relation* ψ *on* S/σ *in the following way:*

For $Q_1,\dots,Q_{n+1} \in S/\sigma$, *if*

a) $|\{Q_1,\dots,Q_{n+1}\}| \neq 2$, *then*

$(Q_1^{n+1}) \in \psi$ *if and only if there are* $x_1,\dots,x_{n+1} \in S$, $x_i \neq x_j$, *for* $i \neq j$, $i,j \in \{1,\dots,n+1\}$, *such that*

I $A_i = \{x_1,\dots,x_{n+1}\} \setminus \{x_i\} \subseteq Q_i$, $i=1,\dots,n+1$, *and that*

II $(x_{i_1},\dots,x_{i_{n+1}}) \in \rho$ *when* $(x_{i_1},\dots,x_{i_{n+1}}) \in A_1 \times \dots \times A_{n+1}$;

and if

b) $|\{Q_1,\dots,Q_{n+1}\}| = 2$, *then*

$(Q_1^{n+1}) \in \psi$ *iff there is exactly one set with* n+1 *element* $\{x_1,\dots,x_{n+1}\} \in S$, $x_i \neq x_j$, *for* $i\neq j$, $i,j \in \{1,\dots,n+1\}$, *such that* I *and* II *hold.*

Then ψ *is* (n+1)-*reflexive,* (n+1)-*antisymmetric, and* $n\bar{A}_1$-*transitive relation on* S/σ.

P r o o f.

a) $(Q_1^{n+1}) \in \psi$ if and only if there are $x_1,\dots,x_{n+1}$, $x_i \neq x_j$, such that A_1 defined in I is a subset of Q_1, and that II is satisfied for $A_i = A_1$, $i=1,\dots,n+1$. Since each class contains at least n elements, A_1 always exists, and II is a consequence of the definition of S/σ. Thus, ψ is (n+1)-reflexive.

b) Let $(Q_{\pi(1)},\dots,Q_{\pi(n+1)}) \in \psi$, for each $\pi \in \{1,\dots,n+1\}!$.

Then for each such π , there is exactly n+1 element $x_1,\dots,x_{n+1}$ such that I and II are satisfied, provided that $Q_1,\dots,Q_{n+1}$ are not all equal. Realy, if $\{Q_1,\dots,Q_{n+1}\}$ consists of only two different classes, then this uniqueness is postulated. Otherwise, suppose that for some $\alpha,\beta \in \{1,\dots,n+1\}!$ $Q_{\alpha(1)},\dots,Q_{\alpha(n+1)})$ determines $x_1,\dots,x_{i-1},x_i,\dots,x_{n+1}$, and $Q_{\beta(1)},\dots,Q_{\beta(n+1)}$ determines $x_1,\dots,x_{i-1},x_i',\dots,x_{n+1}$. Now, for Q_r and Q_s, r, s $\in \{1,\dots,n+1\}\setminus\{i\}$, $Q_r \neq Q_s$, we have

$$|Q_r \cap Q_s| = |\{x_1,\dots,x_i,x_i',\dots,x_{n+1}\}\setminus\{x_r,x_s\}| = n \quad \text{which}$$

means thet $Q_r = Q_s$, contrary to our assumption. So we can consider $x_1,\dots,x_{n+1}$. Each of this elements is in at least one class and thus all permutations of (x_1^{n+1}) are in ρ , i.e. all those classes are equal, proving (n+1)-antisymmetry for ψ .

c) ψ is $n\bar{A}_1$ -transitive: Let $(Q_o^n) \in \psi$, $(Q_1^{n+1}) \in \psi$ satisfy the conditions of $n\bar{A}_1$-transitivity. It means that there are $x_o,\dots,x_n$

and $y_1,\dots,y_{n+1}$, satisfying I and II. By the definition of the sets A_i, $\{x_o,\dots,x_n\}=\{y_1,\dots,y_{n+1}\}$, and we can deduce that $x_i=y_i$ for $i=1,\dots,n$, and $x_o = y_{n+1}$[1]. Now, $n\bar{A}_1$-transitivity for ψ follows directly from the same property of ρ .

EXAMPLE 3. $S = \{1,2,3,4\}$, $n=2$.

$$\rho = d_2 \cup \pi(1,2,3) \cup \{(1,2,4),(2,1,4),(1,3,4),(3,1,4),(3,4,1),(4,3,1),(2,3,4),(3,2,4),(3,4,2),(4,3,2)\}.$$

ρ is $(2,2\bar{A}_1)^*$-RT relation on S.

$\sigma = d_2 \cup \pi(1,2,3)$.

S/σ: $Q_1 = \{1,2,3\}$, $Q_2 = \{1,4\}$, $Q_3 = \{2,4\}$, $Q_4 = \{3,4\}$.

3-reflexive, 3-antisymmetric and $2\bar{A}_1$- transitive relation χ , defined in Theorem 6, is given by:

$$\chi = \{(Q_1,Q_1,Q_1),(Q_2,Q_2,Q_2),(Q_3,Q_3,Q_3),(Q_4,Q_4,Q_4),(Q_1,Q_1,Q_2),(Q_1,Q_1,Q_3),(Q_1,Q_1,Q_4),(Q_3Q_4Q_3),(Q_4,Q_3,Q_3),(Q_4,Q_4,Q_2),(Q_4,Q_2,Q_2),(Q_2,Q_4.Q_2),(Q_4,Q_4,Q_3)\}.$$

3-reflexive, 3-antisymmetric and $2\bar{A}_1$-transitive relation ψ, defined in Theorem 7., is given by:

$$\psi = \{\{(Q_i^3)_{i=1,2,3,4}\} \cup \{(Q_1,Q_2,Q_2),(Q_2,Q_1,Q_2),(Q_1,Q_3,Q_3),(Q_3,Q_1,Q_3),(Q_4,Q_4,Q_1),(Q_1,Q_1,Q_2),(Q_1,Q_1,Q_3),(Q_1,Q_1,Q_4),(Q_4,Q_4,Q_2),(Q_4,Q_2,Q_2),(Q_2,Q_4,Q_2),(Q_4,Q_4,Q_3),(Q_4,Q_3,Q_3),(Q_3,Q_4,Q_3),(Q_4,Q_1,Q_3),(Q_1,Q_4,Q_3),(Q_1,Q_4,Q_2),(Q_4,Q_1,Q_2)\}.$$

[1] The statement holds in ternary case also if we require that $x_1=y_1$ and $x_2=y_2$.

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REZIME

STRUKTURA UOPŠTENIH EKVIVALENCIJA SADRŽANIH U $(2,n\bar{A}_1)$-RT RELACIJAMA

U radu se razmatra jedna klasa generalisanih relacija pretporetka ($(2,n\bar{A}_1)$-RT relacija) i ispituje se struktura u njima sadržanih ekvivalencija. Daju se potrebni i dovoljni uslovi pod kojima je taj parcijalno uredjen skup kompletna mreža. Takodje se pokazuje da se na odgovarajućim particijama tipa n može posmatrati uopšteni poredak, indukovan spomenutim generalisanim pretporetkom.

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ТИПИ БАЗИСОВ ДЛЯ ОДНОЙ МОДИФИКАЦИИ АЛГЕБРЫ ЛОГИКИ

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1. ВВЕДЕНИЕ

Е. Постом (|5|,|7|) подробно изучена структура алгебры логики и ее основные свойства. А.И.Мальцев (|4|) предложил рассматривать алгебру к-значной логики как множество P_k к-значных функций с операциями отождествления аргументов, их перестановки приписывания фиктивного аргумента и суперпозиции. Эта алгебра была названа к-значной алгеброй Поста. Алгебры Поста находят широкие приложения в теории автоматов. В.М. Глушков предложил рассматривать алгебры, которые являются модификациями алгебр Поста и связаны с операцией композиции $\otimes$, определяемой тождеством

$$(f \otimes g)(x_1,x_2,\ldots,x_{m+n-1}) = f(g(x_1,x_2,\ldots,x_n),$$
$$(g(x_2,x_3,\ldots,x_{n+1}),\ldots,g(x_m,x_{m+1},\ldots,x_{m+n-1})) ,$$

где f,g- произвольные m-местная и n-местная функции. Эти алгебры находят приложения при рассмотрении логических структур ЭЦВМ (|1|, |2|).

Г.Е.Цейтлин (|6|) исследовал проблематику функциональной полноты для алгебры булевских функции система операции которой в отличие от алгебр Поста содержит вместо суперпозиции операцию $\otimes$ Г.Е.Цейтлин доказал теорему подобную теореме Поста - Яблонского для алгебры Поста, дающую необходимые и достаточные условия чтобы некоторая система функции являлась системой образующих в ϕ^o .

ОПРЕДЕЛЕНИЕ. Система функций $K \subset \phi^{o}$ называется функционально полной в ϕ^{o} если каждая функция из ϕ^{o} получается из функций системы K путём отождествления аргументов, их перестановки, приписывания фиктивного аргумента и применения операции композиции $\otimes$.

ОПРЕДЕЛЕНИЕ. Конечная полная в ϕ^{o} система функций называется базисом, если никакая из её подсистем не является полной в ϕ^{o}.

Целью настоящей статьи является исследование типов базисов для алгебры ϕ^{o}. Подобную проблему для алгебры логики P_2 исследовал Л.Крнич (|8|).

2. НЕКОТОРЫЕ ОБОЗНАЧЕНИЯ

Дальше нам понадобятся нижеследующие множества:

$$T_{oo} = \{f | f \in \phi^{o},\ f(0,0,\dots,0) = f(1,1,\dots,1) = 0\},$$

$$T_{o1} = \{f | f \in \phi^{o},\ f(x,x,\dots,x) = x\},$$

$$T_{1o} = \{f | f \in \phi^{o},\ f(x,x,\dots,x) = x'\},$$

$$T_{11} = \{f | f \in \phi^{o},\ f(0,0,\dots,0) = f(1,1,\dots,1) = 1\}.$$

Заметим, что каждое из множеств T_{oo}, T_{o1}, T_{1o}, T_{11}, содержит $2^{2^{n}-2}$ функций от n аргументов $x_1, x_2, \dots, x_n$.

$$S = \{f | f \in \phi^{o},\ f(x_1,x_2,\dots,x_n) = f'(x_1',x_2',\dots,x_n')\}$$

т.е. множество всех самодвойственных булевских функций. Число самодвойственных функций от n аргументов равно $2^{2^{n-1}}$

$$M_1 = \{f | f \in \phi^{o},\ \forall(\breve{a},\breve{b})\ \breve{a} \leqslant \breve{b} \Rightarrow f(a) \le f(b)\},$$

где $\breve{a} = (a_1,a_2,\dots,a_n)$, $\breve{b} = (b_1,b_2,\dots,b_n)$ и $\breve{a} \leqslant \breve{b}$ если $a_i \le b_i$ ($i=1,2,\dots,n$), т.е. множество всех изотонных булевских функций.

$$M_2 = \{f | f \in \phi^o, \ f' \in M_1\},$$

т.е. множество всех антитонных булевских функций.

Рассмотрим также следующие классы булевских функций:

$$M = M_1 \cup M_2 \ ,$$

т.е. множество всех монотонных булевских функций;

$$S_1 = T_{o1} \cap S,$$

$$S_2 = T_{1o} \cap S \ .$$

Число функций в каждом из множеств S_1 ,S_2 равно $2^{2^{n-1}-1}$.
Дальше нам понадобятся нижеследующие множества:

$$A = T_{oo} \cup T_{o1} \cup T_{11} \ ,$$

$$B = T_{oo} \cup T_{11} \cup S \ ,$$

$$C = T_{oo} \cup T_{11} \cup M \ ,$$

$$D = T_{o1} \cup T_{1o} \cup \{0,1\}.$$

Для каждого $K \subset \phi^o$, пусть $K' = \phi^o \setminus K$. Тогда имеем:

$$A' = T_{1o} \ ,$$

$$B' = (T_{o1} \cup T_{1o}) \setminus S \ ,$$

$$C' = (T_{o1} \cup T_{10}) \setminus M \ ,$$

$$D' = (T_{oo} \cup T_{11}) \setminus \{0,1\}.$$

Теорема Цейтлина о функциональной полноте в ϕ^o.

Для того чтобы система функций $K \subset \phi^o$ была полной, необходимо и достаточно чтобы ней содержались

- по крайней мере одна функция, не принадлежащая A,
- по крайней мере одна функция, не принадлежащая B,
- по крайней мере одна функция, не принадлежащая C,
- по крайней мере одна функция, не принадлежащая D.

Из этой теоремы вытекает, что из всякой полной системы функций в ϕ^o можно выбрать полную подсистему, состоящую не более чем из четырёх функций.

3. ТИПЫ БАЗИСОВ АЛГЕБРЫ Φ^{o}

Если функция f принадлежит классу $X (X \in \{A,B,C,D\})$, будем говорить, что она имеет свойство X . Если функция обладает, например, свойствами A, C и не обладает свойствами B, D, будем говорить что это функция типа A, C и обозначним её через /A,C/. Две функции, обладающие одними и теми же свойствами будем называть функциями одного и того же типа. Функция /ø/ не обладает ни одним из свойств A,B,C,D.

ТЕОРЕМА 1. Число различных типов функций из Φ^{o} равно 9:

1. /A,B,C,D/,	4. /A,C,D/,	7. /B,D/,
2. /A,B,C/,	5. /B,C,D/,	8. /C,D/,
3. /A,B,D/,	6. /A,D/,	9. /D/.

ДОКАЗАТЕЛЬСТВО

I /A,B,C,D/; множество функций этого типа:

$A \cap B \cap C \cap D = (A \cap B) \cap (C \cap D) = (T_{oo} \cup T_{11} \cup (T_{o1} \cap S)) \cap M =$
$= (S_1 \cap M_1) \cap \{0,1\}$.

II /A,B,C/; множество функций этого типа:

$A \cap B \cap C \cap D' = (A \cap B) \cap (C \cap D') = (T_{oo} \cup T_{11} \cup S) \cap ((T_{oo} \cup T_{11}) \setminus \{0,1\}) =$
$= (T_{oo} \cup T_{11}) \setminus \{0,1\}$; такая, например, функция $f(x_1,x_2,x_3) = x_1 + x_2$.

III /A,B,D/; множество функций этого типа:

$A \cap B \cap C' \cap D = (A \cap B) \cap (C' \cap D) = (T_{oo} \cup T_{11} \cup (T_{o1} \cap S)) \cap ((T_{o1} \cup T_{1o}) \setminus M) =$
$= (T_{o1} \cap S) \setminus M = S_1 \setminus M_1$; такая, например, функция $f(x_1,x_2,x_3) =$
$= x_1' x_2 \vee x_1' x_3 \vee x_1 x_2 x_3$.

IV /A,C,D/; множество функций этого типа:

$A \cap B' \cap C \cap D = (A \cap C) \cap (B' \cap D) = (T_{oo} \cup T_{11} \cup M_1) \cap ((T_{o1} \cup T_{1o}) \setminus S) =$
$= M_1 \setminus (\{0,1\} \cup S_1)$; такая, например, функция $f(x_1,x_2,x_3) = x_1 \vee x_2$.

V /B,C,D/; множество функций этого типа:

$A' \cap B \cap C \cap D = (A' \cap D) \cap (B \cap C) = T_{10} \cap (T_{00} \cup T_{11} \cup (M \cap S)) =$

$= S_2 \cap M_2$; такая, например, функция $f(x_1, x_2, x_3) = x_1'$.

VI /A,B/; такая функция не существует, потому что

$A \cap B \cap C' \cap D' = (A \cap B) \cap (C' \cap D') = (A \cap B) \cap ((T_{01} \cap T_{10}) \setminus M) \cap$

$\cap ((T_{00} \cup T_{11}) \setminus \{0,1\}) = (A \cap B) \cap \emptyset = \emptyset$

VII /A,C/; такая функция не существует, потому что

$A \cap B' \cap C \cap D' = (A \cap C) \cap (B' \cap D') = (A \cap C) \cap ((T_{01} \cup T_{10}) \setminus S) \cap$

$\cap ((T_{00} \cup T_{11}) \setminus \{0,1\}) = (A \cap C) \cap \emptyset = \emptyset$.

VIII /A,D/; множество функций этого типа:

$A \cap B' \cap C' \cap D = (A \cap B') \cap (C' \cap D) = (T_{01} \setminus S_1) \cap ((T_{01} \cup T_{10}) \setminus M) =$

$= T_{01} \setminus (S_1 \cup M_1)$; такая, например, функция $f(x_1, x_2, x_3) =$

$= x_1' x_2' x_3 \vee x_1 x_2 x_3$.

IX /B,C/; такая функция не сушествует, потому что

$A' \cap B \cap C \cap D' = (A' \cap D') \cap (B \cap C) = \emptyset \cap (B \cap C) = \emptyset$.

X /B,D/; множество функций этого типа:

$A' \cap B \cap C' \cap D = (A' \cap B) \cap (C' \cap D) = (T_{10} \cap S) \cap ((T_{01} \cup T_{10}) \setminus M) =$

$= S_2 \setminus M_2$; такая, например, функция $f(x_1, x_2, x_3) = x_1' x_2 x_3 \vee x_1 x_2' \vee x_1 x_3'$.

XI /C,D/; множество функций этого типа:

$A' \cap B' \cap C \cap D = (A' \cap B') \cap (C \cap D) = (T_{10} \setminus S_2) \cap M = M_2 \setminus (\{0,1\} \cup S_2)$;

такая, например, функция $f(x_1, x_2, x_3) = x_2' \vee x_3'$.

XII /A/; такая функция не сушествует, потому что

$A \cap B' \cap C' \cap D' = (A \cap B') \cap (C' \cap D') = (A \cap B') \cap \emptyset = \emptyset$.

XIII /B/; такая функция не сушествует, потому что

$A' \cap B \cap C' \cap D' = (A' \cap B) \cap (C' \cap D') = (A' \cap B) \cap \emptyset = \emptyset$.

XIV /C/; такая функция не сушествует, потому что

$A' \cap B' \cap C \cap D' = (A' \cap C) \cap (B' \cap D') = (A' \cap C) \cap \emptyset = \emptyset$.

XV /D/; множество функций этого типа:

$A' \cap B' \cap C' \cap D = (A' \cap D) \cap (B' \cap C') = T_{10} \cap ((T_{01} \cup T_{10}) \setminus$
$\setminus (S \cup M)) = T_{10} \cap (S_2 \cup M_2)$; такая, например, функция $f(x_1, x_2, x_3) =$
$= x_1' x_2' \vee x_1 x_3'$.

XVI /∅/; такая, функция не существует, потому что
$A' \cap B' \cap C' \cap D' = (A' \cap D') \cap (B' \cap C') = \emptyset \cap (B' \cap C') = \emptyset$

Теорема доказана.

Будем составлять базисы, выражая их через функции 1-9. Для построения базисов полезны нижеследующие замечания;

Замечание 1. Функция /A,B,C,D/ не содержится ни в одном базисе, потому что принадлежит всем классам A,B,C,D.

Замечание 2. Одночленные базиси не существуют, потому что не существуют функции типа /∅/.

Замечание 3. Для построения трёхчленных базисов не учитываем функций /D/ .

Замечание 4. Для построения четырёхчленных базисов не учитываем функции /D/, /A,D/, /B,D/, /C,D/.

Теперь легко найдутся все типы базисов в ϕ^{o}.

I Двухчленные типы базисов:

{/A,B,C/,/D/}.

II Трёхчленные типы базисов:

{/A,B,C/,/C,D/,/A,B,D/} ; {/A,B,C/,/C,D/,/A,D/} ;
{/A,B,C/,/C,D/,/B,D/} ; {/A,B,C/,/B,D/,/A,C,D/} ;
{/A,B,C/,/B,D/,/A,D/} ; {/A,B,C/,/A,D/,/B,C,D/} .

III Четырёхчленные типы базисов:

{/A,B,C/,/A,B,D/,/A,C,D/,/B,C,D/}.

Таким образом, доказана следующая теорема:

ТЕОРЕМА 2. Число различных типов базисов в ϕ^0 равно 8: 1 двухчленный тип, 6 трёхчленных типов и 1 четырёхчленный тип.

Замечание 5. Функции $x_1, x_2, \dots, x_n$ и константы 0 и 1 не содержатся ни в одном базисе.

Замечание 6. В каждом базисе содержится точно одна функция f для которой $f(0,0,\dots,0) = f(1,1,\dots,1)$.

Пусть $t_n/A/$ обозначает число всех n-местных функции типа /A/, $t_n/B,C/$ число всех n-местных функции типа /B,C/ в ϕ^0, и т.п. Легко получается что

$$t_n/A,B,C/ = 2^{2^n-1}-2,$$

$$t_n/A,B,D/ < 2^{2^{n-1}-1},$$

$$t_n/A,C,D/ < 2^{2^n-2},$$

$$t_n/B,C,D/ < 2^{2^{n-1}-1},$$

откуда получается следующее неравенство:

$$N_n^4 < 2^{3\cdot 2^n-5},$$

где N_n^4 обозначает число чётырёхчленных базисов в ϕ^0, состоящихся только из n-местных функций.

Заметим что

$$\lim_{n\to\infty} \frac{N_n^4}{\binom{2^{2^n}}{4}} = 0,$$

где $\binom{2^{2^n}}{4}$- число всех четырёхчленных систем n-местных функций в ϕ^0.

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REZIME

TIPOVI BAZA ZA JEDNU MODIFIKACIJU ALGEBRE LOGIKE

U radu se ispituju medjusobni odnosi medju maksimalnim podalgebrama algebre ϕ^0 (jedne modifikacije algebre logike). Posmatra se relacija ekvivalencije takva da su dve funkcije f i g ekvivalentne tj. da su istog tipa ako za svaku maksimalnu podalgebru X od ϕ^0 važi tačno jedna od sledeće dve mogućnosti:

1° $f \in X$ i $g \in X$,

2° $f \notin X$ i $g \notin X$.

TEOREMA 1. Ima tačno 9 različitih tipova funkcija u ϕ^0.

Pomoću takvih funkcija izgradjuju se baze algebre ϕ^0, pri čemu se koristi teorema Cejtlina (|6|). Na taj način, svaka baza

albegre ϕ^{o} pripada odredjenom tipu baza.

TEOREMA 2. Postoji tačno 8 različitih tipova baza u algebri ϕ^{o} : 1 tip dvočlanih, 6 tipova tročlanih i 1 tip četvoročlanih baza.

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NOTE ON THE SPANNING TREES OF A CONNECTED DIGRAPH

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ABSTRACT

Our aim in this paper is to give some relations between the spanning trees and some determinants obtained from the incidence matrix of a connected digraph. The spanning trees that differ by one edge are also investigated.

Let $D=(V,E)$ be a connected <u>digraph</u> (directed graph) with $V=\{v_1,v_2,\dots,v_p\}$ the set of <u>vertices</u>, $E=\{e_1,e_2,\dots,e_q\}$ the set of <u>edges</u>, and $S=(s_{ij})$, $i=1,2,\dots,p$; $j=1,2,\dots,q$, the <u>incidence martix</u>, where

$$s_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is the initial vertex of } e_j, \\ -1, & \text{if } v_i \text{ is the terminal vertex of } e_j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{S}$ be the matrix obtained from S by deleting the line corresponding to the vertex v_p. If $T=\{e_{j_1},e_{j_2},\dots,e_{j_m}\}$ $(m=p-1)$ is a spanning tree of D, we shall denote by $\bar{S}(T)$ the square submatrix of $\bar{S}$ obtained with the lines of $\bar{S}$ and the columns $j_1,j_2,\dots$ $\dots,j_m$. Because T is a spanning tree, there exists a unique chain connecting any two vertices in the graph (V,T). Let $c_i(v_i,v_p)$ such a chain connecting v_i with v_p, $i=1,2,\dots,m$, and $E(c_i)$ the

edge-set of c_i. Let $e_{\alpha(i)}$, $\alpha(i) \in \{j_1, j_2, \ldots, j_m\}$ the unique edge incident with the vertex v_i, $i=1,2,\ldots,m$, for which $e_{\alpha(i)} \in E(c_i) \cap T$, and $\varepsilon(i) \in \{1,2,\ldots,m\}$ such that $\alpha(i) = j_{\varepsilon(i)}$.

We consider the matrix $\bar{\bar{S}}(T) = [\bar{\bar{S}}(T)]_{i\beta}$, $i,\beta=1,2,\ldots,m$, where

$$[\bar{\bar{S}}(T)]_{i\beta} = \begin{cases} 0, & \text{if } \beta \neq \varepsilon(i), \\ [\bar{S}(T)]_{i,\varepsilon(i)}, & \text{if } \beta = \varepsilon(i). \end{cases}$$

THEOREM 1. $\det[\bar{S}(T)] = \det[\bar{\bar{S}}(T)]$.

P r o o f. We denote by $S(T)$ the submatrix of S obtained with the lines od S and the columns $j_1, j_2, \ldots, j_m$.

Let v_{t_1} ($v_{t_1} \neq v_p$) a terminal vertex of $T^{(1)} = T$. Adding the line t_1 of the matrix $S(T)$ to the line corresponding to the other vertex of $e_{\alpha(t_1)}$ we obtain the matrix $S_1(T)$. We consider now the tree $T^{(2)}$ obtained from $T^{(1)}$ by deleting the vertex v_{t_1} and the edge $e_{\alpha(t_1)}$. Let v_{t_2} ($v_{t_2} \neq v_p$) a terminal vertex of $T^{(2)}$. Adding the line t_2 of the matrix $S_1(T)$ to the line corresponding to the other vertex of $e_{\alpha(t_2)}$ we obtain the matrix $S_2(T)$. Repeating the above thus give rise to the matrix $S_m(T)$. For this matrix the p-th line is null.

Denoting by $\bar{S}_k(T)$, $k=1,2,\ldots,m$, the matrix obtained from $S_k(T)$ by deleting the p-th line, then $\bar{\bar{S}}(T) = \bar{S}_{p-1}(T)$.

On the other hand, according to the properties of determints we have

$$\det[\bar{\bar{S}}(T)] = \det[\bar{S}_{p-1}(T)] = \det[\bar{S}_{p-2}(T)] = \ldots = \det[\bar{S}_1(T)] =$$

$\det[\bar{S}(T)]$, and the theorem is proved.

Let T_1 and T_2 two spanning trees of D. By |1| and theorem 1 it follows that

$$\det[\bar{\bar{S}}(T_1)] = \pm 1 \ ,$$
(1)
$$\det[\bar{\bar{S}}(T_2)] = \pm 1 \ .$$

Obviously, in $\bar{\bar{S}}(T_1)$ and $\bar{\bar{S}}(T_2)$ each line and each column contains a single nonull element (equal to ± 1). For an arbitrary column, if we want to have on the same place the nonull element of $\bar{\bar{S}}(T_2)$ as in $\bar{\bar{S}}(T_1)$, we must permute two columns in $\bar{\bar{S}}(T_2)$. Let π the total number of permutations necessary for all nonull elements of $\bar{\bar{S}}(T_2)$.

Let σ the total number of exchanges of sing such that each nonull element of $\bar{\bar{S}}(T_2)$ in the same place as in $\bar{\bar{S}}(T_1)$, to have the same sign

But, every permutation and every exchange of sing multiplies the value of $\det[\bar{\bar{S}}(T_2)]$ by -1. Hence, by (1) we have

(2) $$\det[\bar{\bar{S}}(T_2)] = (-1)^{\pi+\sigma} \det[\bar{\bar{S}}(T_1)] \ .$$

By (2) and theorem 1 it follows that

(3) $$\det[\bar{S}(T_1)] \det[\bar{S}(T_2)] = (-1)^{\pi+\sigma} \ .$$

Let T_1 and T_2 two spanning trees of D such that $|T_1 - T_2| = k$. Deleting the k distinct edges, every spanning tree becomes a graph containing k+1 connected components. Moreover, the k+1 connected components in T_1 and T_2 are identical, and only one of them contains the vertex v_p. The k components that do not contain v_p are called <u>principal</u>.

Obviously, every vertex of a principal component is connected by a unique chain with v_p in (V,T_i), i=1,2, and every chain (one of T_1 and other of T_2) contains an unique edge (one of T_1-T_2 and other of T_2-T_1) incident with the principal component. We call these edges <u>principal</u>.

If the principal edges have the same orientation related to the principal component, then this component is <u>positive</u> and <u>negative</u> otherwise.

We consider the graph having as vertices the k+1 components and as edges the principal edges (from T_1 and T_2) incident to the above components. We denote by $\sigma(T_1,T_2)$ the number of positive components from which we subtract the number of cycles in the graph above considered. By |2| we have

$$(4) \quad \det[\bar{S}(T_1)]\ \det[\bar{S}(T_2)] = (-1)^{\sigma(T_1,T_2)} .$$

Let T_1,T_2 two spanning trees of D for which $T_1-T_2=\{a\}$ and $T_2-T_1=$ $=\{b\}$, $a\neq b$.

We denote by $\omega(T_1,b)$ the unique cycle contained in $(V,T_1 \cup$ $\cup\{b\})$.

THEOREM 2.

$$\det[\bar{S}(T_1)]\det[\bar{S}(T_2)] = \begin{cases} -1, & \textit{if a and b have the same orientation in } \omega(T_1,b), \\ 1, & \textit{otherwise}. \end{cases}$$

P r o o f. Deleting the edge a from T_1 we obtain a graph that contains two connected components; one of them contains the vertex v_p and the other is principal.

Obviously, a and b are principal edges. If a and b have the same orientation in $\omega(T_1,b)$, then the principal component is negative, i.e., $\sigma(T_1,T_2)=-1$. This, by (4), it follows that

$$\det[\bar{S}(T_1)]\det[\bar{S}(T_2)] = -1 .$$

If a and b have not the same orientation in $\omega(T_1,b)$, then the principal component is negative, i.e., $\sigma(T_1,T_2)=0$. Then by (4) it follows that $\det[\bar{S}(T_1)]\det[\bar{S}(T_2)]=1$, and the theorem is proved. Let $T=\{e_{j_1},e_{j_2},\dots,e_{j_m}\}$, $(m=p-1)$ a spanning tree of D. Deleting from T the edge e_{j_h} $(1\leq h\leq m)$ we obtain two connected components V_h and $\bar{V}_h$. .

To the bipartition $(V_h,\bar{V}_h)$ we can associate a cocycle $C(e_{j_h},T)$ that contains the edge e_{j_h}.

Obviously, if T_1 and T_2 are two spanning trees for which $T_2 - T_1 = \{b\}$ and $T_1 - T_2 = \{a\}$, then $b \in C(a, T_1)$. Moreover, if $c \in C(a, T_1)$, then $(T_1 - \{a\}) \cup \{c\}$ is a spanning tree.

Let T_o a spanning tree and $T_1, T_2, \ldots, T_r$ all spanning trees for which $T_o - T_k = \{a_o\}$ and $T_k - T_o = \{a^{(k)}\}$, $k=1,2,\ldots,r$. Thus $C(a_o, T_o) = \{a_o, a^{(1)}, \ldots, a^{(r)}\}$.

Moreover, if $b \in C(a_o, T_o)$, then $(T_o - \{a_o\}) \cup \{b\}$ is one of $T_1, T_2, \ldots, T_r$.

Let $A(T_o, a_o) = \{T_1, T_2, \ldots, T_r\}$. Obviously, we have

$$A(T_o, a_o) = \bigcup_{\substack{b \in C(a_o, T_o) \\ b \neq a_o}} \{(T_o - \{a_o\}) \cup \{b\}\} .$$

If $T_o = \{e_{j_1}, e_{j_2}, \ldots, e_{j_m}\}$, then every spanning tree T with $|T_o - T| = 1$ belongs to one of $A(T_o, e_{j_h})$, $h=1,2,\ldots,m$. Also, all spanning trees of $A(T_o, e_{j_h})$ are distinct. Indeed, each $T \in A(T_o, e_{j_t})$ does not contain the edge e_{j_t}.

On the other hand, for every $T \in A(T_o, e_{j_t})$ holds $|T_o - T| = 1$, i.e., all edges of T_o (except for e_{j_t}) belong to T. Hence, T does not belong to $A(T_o, e_{j_s})$ with $t \neq s$.

Let $v_o \in V$ and $e_o \in E$ arbitrary choosen, such that e_o is incident with the vertex v_o. We denote by $C(v_o)$ the cocycle associated to the bipartition $(\{v_o\}, V - \{v_o\})$.

Let $A(e_o)$ the set of spanning trees that contain the edge e_o and $\bar{A}(e_o)$ the set of spanning trees that do not contain e_o.

THEOREM 3.

$$(5) \quad \bar{A}(e_o) = \bigcup_{\substack{T \in A(e_o) \\ b \in C(e_o, T) \cap C(v_o)}} \{(T - \{e_o\}) \cup \{b\}\} .$$

P r o o f. Obviously, every spanning tree obtained by (5) belongs to $\bar{A}(e_o)$. Suppose now there exists $T \in \bar{A}(e_o)$ such that it cannot be obtained by (5).

Let $\omega(T,e_o)$ the unique cycle contained in the graph $(V, T \cup \{e_o\})$. If $b (b \neq e_o)$ is an edge of $\omega(T,e_o)$ incident with the vertex v_o, then the spanning tree $T' = (T-\{b\}) \cup \{e_o\}$ belongs to $A(e_o)$. But $b \in C(e_o,T') \cap C(v_o)$, i.e., T can be obtained from T′ by (5); contradiction. Hence each spanning tree of $\bar{A}(e_o)$ can be obtained by (5) and the theorem is proved.

THEOREM 4. *Every element of* $\bar{A}(e_o)$ *is obtained only once by (5).*

P r o o f. Suppose that T is a spanning tree of $\bar{A}(e_o)$ often generated by (5). In this case there exist at least two distinct edges c and d in T incident with the vertex v_o such that $(T-\{c\}) \cup \{e_o\}$ and $(T-\{d\}) \cup \{e_o\}$ are distinct elements of $A(e_o)$, i.e., c and d belong to the unique cycle $\omega(T,e_o)$. This is impossible. Hence the theorem is true.

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REZIME

NOTA O POKRIVAJUĆIM STABLIMA
ORIJENTISANOG DIGRAFA

U ovom radu se ispituju odnosi izmedju pokrivajućih stabala orijentisanog digrafa i determinanata nekih podmatrica matrice incidencije toga digrafa. Takodje su ispitivani parovi pokrivajućih stabala digrafa koja se razlikuju u orijentaciji samo jedne grane.

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The journal "Zbornik radova Prirodno-matematičkog fakulteta. Novi Sad. Serija za matematiku" (Review of Research, Faculty of Science. Novi Sad. Mathematics series) (in the following "Zbornik") first appeared in 1971 as the official publication of the Institute of Mathematics of the Faculty of Science, University of Novi Sad.

In Novi Sad the Department of Mathematics was founded in 1954 as a part of the Faculty of Arts and Natural Science, University of Novi Sad, and members of the department had been publishing their scientific contributions in "Godišnjak Filozofskog fakulteta u Novom Sadu" (Annuaire de la Faculté des lettres et sciences, Novi Sad) in volumes from I (1956) to XII (1969).

From 1969 the Department of Mathematics belongs to the Faculty of Science. In 1976 the Department of Mathematics became Institute of Mathematics.

The number of papers which appear in "Zbornik" increases regularly.

Among the reviewers of papers which appear in "Zbornik" are known yugoslav and foreign mathematicians. This provides the appropriate level of accepted papers. A diversification of problems which are dealt with in "Zbornik" can be noticed.

From volume 11(1981) all papers will be published in one of the following world languages: English, Russian, German, French.

All published papers are reviewed regularly in: Mathematical Reviews; Реферативный журнал 13. Математика. Сводный том; Zentrallblat für Mathematik und ihre Grenzgebiete, Mathematics Abstracts.

The increase of "Zbornik´s" volume was followed by an increase in exchange with other similar institutions. At present "Zbornik" is exchanged with 295 foreign and 45 yugoslav institutions. Among the publications which are obtained for "Zbornik" are many known international journals.

The enclosed "Bibliography" contains a chronological list of titles which appeared in "Zbornik" from volume 1(1971) to (10(1980). In the appendix the Author Index and Subject Classification are given.

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